① <u>Recall</u>: derived Fukaya cat.: $\mathcal{A}$ A_∞-cat. $\rightsquigarrow$ $Tw\,\mathcal{A}$ triangulated A_∞-cat. of twisted complexes $\rightsquigarrow$ $D^b(\mathcal{A}) = H^0(Tw\,\mathcal{A})$ honest tri. cat.

- <u>$Tw\,\mathcal{A}$</u>: objects = <u>twisted complexes</u> (X, δ_X), $X = \bigoplus_{i=1}^{r} X_i[k_i]$ $\quad X_i \in ob\,\mathcal{A}$, $k_i \in \mathbb{Z}$

$$\delta_X = (\delta_X^{ij}) \in hom^1(X,X) \text{ degree 1 endomorphism,}$$
$$\delta_X^{ij} \in hom_{\mathcal{A}}(X_i, X_j) \text{ of degree } k_j - k_i + 1$$

δ_X strictly lower-triangular, $\sum_{k\geq 1} m_k(\delta_X \dots \delta_X) = 0$ (generalizes $\delta_X^2 = 0$)

<u>morphisms</u> = $hom((X,\delta_X),(Y,\delta_Y)) = \bigoplus_{i,j} hom(X_i, Y_j)[l_j - k_i]$, where $X = \bigoplus X_i[k_i]$, $Y = \bigoplus Y_j[l_j]$

given $k+1$ twisted complexes $X_0 \dots X_k$ & maps a_i b/w them,

$$m_k^{Tw}(a_k, \dots, a_1) = \sum_{i_0 \dots i_k} m_{k+i_0+\dots+i_k}(\underbrace{\delta_{X_k} \dots \delta_{X_k}}_{i_k}, a_k, \dots, a_1, \underbrace{\delta_{X_0} \dots \delta_{X_0}}_{i_0})$$

E.g. $m_1^{Tw}(a) = m_1(a) + m_2(\delta_Y, a) + m_2(a, \delta_X) + \dots$
$a: X \to Y$

(generalizes: diff^l on hom's of complexes)

$m_2^{Tw}(a_2, a_1) = m_2(a_2, a_1) + \dots$

☆ $Tw\,\mathcal{A}$ is a triangulated A_∞-category ($\exists$ mapping cones, like usual complexes)

- <u>derived category</u> $D(\mathcal{A}) := H^0(Tw\,\mathcal{A})$: same objects, but
$hom(X,Y) := H^0(hom^{Tw\,\mathcal{A}}(X,Y), m_1^{Tw\,\mathcal{A}})$ (NB: $hom(X, Y[k]) = H^k(\dots)$)
[analogue of: chain maps up to homotopy]
composition = induced by $m_2^{Tw\,\mathcal{A}}$ on cohomology.

<u>Rmk</u>: no need to localize wrt quasi-isoms, in an A_∞-category quasi-isos are already invertible up to homotopy.

- <u>Variant: split-closed der. cat.</u>
$X \in \mathcal{A}$ linear cat., $p \in Hom_{\mathcal{A}}(X,X)$ is idempotent if $p^2 = p$.
Image of p := Y + maps $X \underset{v}{\overset{u}{\rightleftarrows}} Y$ s.t. $uv = id_Y$, $vu = p$

doesn't always exist in $\mathcal{A}$ $\Rightarrow$ need enlargement to achieve this.

Split-closure of $\mathcal{A}$: objects $= (X, p)$, p idempotent endom. of X

$$\hom((X,p),(Y,p')) = p' \hom(X,Y)\, p$$

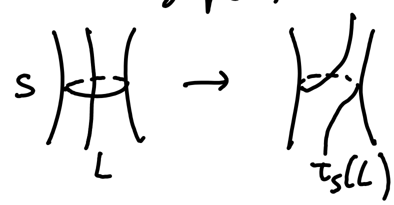
In A_∞ setting, use a more sophisticated approach (Yoneda embedding to A_∞-modules, modules which are quasi-isom. to abstract image of an idempotent).

Geometrically:

- some exact triangles in derived Fukaya category can be understood as Lagr. connected sum / Dehn twist [Seidel, see also FOOO]

Ex.: S Lagrangian sphere $\rightsquigarrow$ τ_S Dehn twist $\in \mathrm{Symp}(M,\omega)$

easiest in 1-dim'l case:



L Lagr. $\rightsquigarrow$ $\tau_S(L)$ Lagrangian

(in higher dim., defined using geodesic flow in nbd of $S \simeq T^*S$)

Seidel: $\exists$ exact triangle in $D\mathrm{Fuk}(M)$:

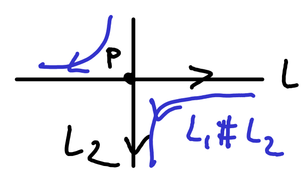
$$\begin{tikzcd} HF^*(S,L) \otimes S \arrow[rr, "ev"] & & L \arrow[dl] \\ & \tau_S(L) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

($\leftrightarrow$ long exact sequence for $HF(L',-)$)

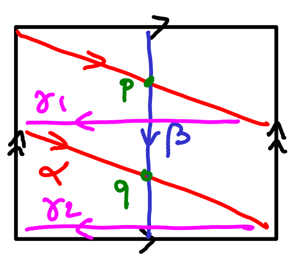
FOOO: L_1, L_2 graded Lagrangian, $L_1 \cap L_2 = p$ of index 0

$\rightsquigarrow$ $L_1 \#_p L_2 \simeq \mathrm{Cone}(L_1 \xrightarrow{p} L_2)$

vs. "$L_1[1] \cup_p L_2 \simeq \mathrm{Cone}(L_1 \xrightarrow{0} L_2)$"



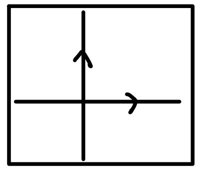
- So e.g. consider T^2:



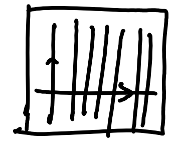
$\mathrm{Cone}(\alpha \xrightarrow{p+q} \beta)$

$\simeq$ disconnected Lagrangian $\gamma_1 \oplus \gamma_2$

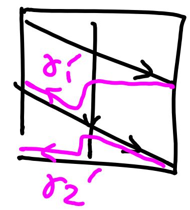
If we only started with α & β, der. cat. would have $\gamma_1 \oplus \gamma_2$ but not γ_1 & γ_2 separately; split-closure addresses this.

③ ▸ If we start with 2 generators  successive Dehn twists give all homotopy classes of loops on T^2;

but each homotopy class ∋ ∞ly many non-Ham.-isotopic Lagrangians.

To generate $D\mathrm{Fuk}(T^2)$ as triangulated envelope we need e.g. 1 horiz. loop + ∞ly many vertical loops 

On the other hand, α & β as above <u>split generate</u>.

key point: $\mathrm{Cone}(\alpha \xrightarrow[p+T^a q]{} \beta)$ gives 

direct sums of loops that vary continuously within homotopy class

▸ But many cones & idempotents don't have an obvious geom interpretation.

e.g. Clifford torus $T = \{|x|=|y|=|z|\} \subset \mathbb{CP}^2$ has idempotents $\in HF(T,T)$ without any obvious geometric interpretation.

<u>Homological mirror symmetry conjecture</u> (Kontsevich 1994)

$X, X^\vee$ mirror Calabi-Yaus $\Rightarrow$ $D^\pi\mathrm{Fuk}(X) \simeq D^b\mathrm{Coh}(X^\vee)$ and vice versa.

Let's see how this works <u>at homology level</u> in example of T^2 (after Polishchuk - Zaslow)

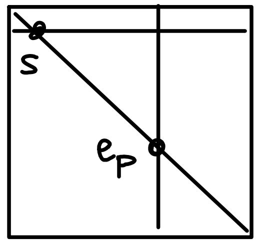
Consider on symplectic side $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\omega = \lambda\, dx\wedge dy$ $(\int_{T^2}\omega = \lambda)$

on complex side $X^\vee = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, $\tau = i\lambda$.

- Lagrangian submflds in X can be isotoped (Hamiltonianly) to straight lines with rational slope; likewise, we can arrange flat conn. 1-form to be constant.

We'll see: family of Lagrangians in homology class (p,q) (+ $U(1)$ conn.) $\longleftrightarrow$ family of rank p vector bundles on $X^\vee$ with $c_1 = -q$.

④ Want to test:



$L_0 \leftrightarrow \mathcal{O}$

Calculate: $m_2(s, e_p) = ?$

$L_1 \leftrightarrow \mathcal{L}$ degree 1 line bundle

$L_p \leftrightarrow \mathcal{O}_p$ skyscraper sheaf of some point p

mirror to: $\mathcal{O} \xrightarrow[s]{} \mathcal{L} \xrightarrow[ev_p]{} \mathcal{O}_p$ composition = value $s(p)$ of s at p.

s: holom. section of $\mathcal{L}$

ev_p: evaluation at p (in some trivialisation of $\mathcal{L}_{|p}$!!)

$\Rightarrow$ first need to understand: formulas for sections of holom. line bundles over the elliptic curve $X^\vee$? $\Rightarrow$ THETA FUNCTIONS.

- $L \to X^\vee$ line bundle $\rightsquigarrow$ pullback to univ. cover $\mathbb{C} \to X^\vee$ is holomorphically trivial; in fact, pullback to $\mathbb{Z}$-cover $\mathbb{C}/\mathbb{Z}$ too.

$$L \simeq \mathbb{C} \times \mathbb{C} \Big/ \begin{array}{l} (z,v) \sim (z+1, v) \\ (z,v) \sim (z+\tau, \varphi(z) v) \end{array}$$

where φ holomorphic, $\varphi(z+1) = \varphi(z)$

Ex: $\varphi(z) = e^{-2\pi i z} e^{-\pi i \tau}$ determines degree 1 line bundle $\mathcal{L}$ with section the theta function $\theta(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2\pi i \left(\frac{\tau n^2}{2} + nz\right)}$

generalization: $$\theta[c', c''](\tau, z) := \sum_{m \in \mathbb{Z}} \exp 2\pi i \left(\frac{\tau (m+c')^2}{2} + (m+c')(z+c'') \right)$$

satisfies
$$\theta[c', c''](\tau, z+1) = e^{2\pi i c'} \theta[c', c''](\tau, z)$$
$$\theta[c', c''](\tau, z+\tau) = e^{-\pi i \tau} e^{-2\pi i (z+c'')} \theta[c', c''](\tau, z)$$

$$\left[\frac{\tau (m+c')^2}{2} + \tau(m+c') + (m+c')(z+c'') = \frac{\tau (m+1+c')^2}{2} - \frac{\tau}{2} + (m+1+c')(z+c'') - (z+c'') \right]$$

- sections of $\mathcal{L}^{\otimes n}$ are $\theta\left[\frac{k}{n}, 0\right](n\tau, nz)$. $k = 0, \dots, n-1$
- other line bundles are obtained by pullback by translation $z \mapsto z + c''$ i.e. $\theta[0, c'']$ etc...