Recall: derived Fukaya cat.: $A_{\text{Aoo-cat}} \to TwA$ triangulated $A_{\text{oo-cat}}$ of twisted complexes $\to D^b(A) = H^0(TwA)$ honest bi-cat.

- $TwA$: objects = twisted complexes $(X, S_X)$, $X = \oplus_{i=1}^{\infty} X_i[k_i]$ $x_i \in \text{ob } A$, $k_i \in \mathbb{Z}$
  
  $S_X = (S^i_X) \in \text{hom}^4(X, X)$ degree $1$ endomorphism,
  
  $S^i_X \in \text{hom}_A(X_i, X_j)$ of degree $k_j - k_i + 1$

  $S_X$ strictly bim-triangular, $\sum_{k \geq 1} m_k(S_X \ldots S_X) = 0$

  (generalizes $S_X^2 = 0$)

  morphisms $= \text{hom}((X, S_X), (Y, S_Y)) = \bigoplus_{i,j} \text{hom}(X_i, Y_j)[b_j - k_i]$

  $\oplus X_i[k_i] \oplus Y_j[b_j]$

  given $k+1$ twisted complexes $X_0 \ldots X_k$ & maps $a_i$: b/w them,

  $m^Tw_k(a_0, \ldots, a_k) = \sum_{i_0 \ldots i_k} m_{k+i_0 + \ldots + i_k} (S_{x_{i_0}} \ldots S_{x_{i_k}}, a_{i_0}, \ldots, a_k, S_{x_{i_k}} \ldots S_{x_0})$

  E.g. $m^Tw^k(a) = m_1(a) + m_2(S_y, a) + m_2(a, S_x) + \ldots$

  $a: X \to Y$

  (generalizes: diff. on hom's of complexes)

  $m^Tw^2(a_2, a_1) = m_2(a_2, a_1) + \ldots$

- $TwA$ is a triangulated $A_{\text{oo-cat}}$ (3 mapping cones, like usual complexes)

- derived category $D(A) := H^0(TwA)$: same objects, but

  $\text{hom}(X, Y) := H^0(\text{hom}^{Tw}(X, Y), m^Tw^1)$

  (where $\text{hom}(X, Y[k]) = H^k(\ldots)$)

  [analogue of: chain maps up to homotopy]

  composition = induced by $m^Tw^2$ on cohomology.

  Remark: no need to localize wrt quasi-isos, in an $A_{\text{oo-cat}}$ quasi-isos are already invertible up to homotopy.

- Variant: split-closed deriv. cat.

  $x \in A$ linear cat., $p \in \text{Hom}_A(x, x)$ idempotent if $p^2 = p$

  Image of $p := Y + \text{im } p \xrightarrow{\text{uv}} Y$ s.t. $uv = id_Y$, $vu = p$
doesn't always exist in $\mathfrak{a}=\mathbb{C}$ need enlargement to achieve this.

Split closure of $\mathfrak{a}$: objects $= (X,p)$, $p$ identifies class of $X$
 hom $(X,p), (Y,p') = p \text{ hom}(X,Y)p$

In $\mathfrak{a}^\infty$ setting, use a more sophisticated approach
( Yoneda embedding to $\mathfrak{a}^\infty$-module, module with $\mathfrak{a}^\infty$-action to abstract image of an idele class).

Geometrically:

- Some exact triangles in derived Fukaya category can be understood as
  Lagrangian connected sum / Dehn twist [Seidel; see also FOOO]

**Ex:** $S$ Lagrangian sphere $\sim T^2$ Dehn twist $E_{S_y}(M,\xi)$

  (in higher dim, defined using geodesic flow in $\text{Mod}_S \cong T^*S$)

  *exact triangle in $\mathcal{D}$Fuk($M$):* $\text{HF}^*(S,L) \otimes S \rightarrow L$

  (long exact sequence for $HF(L',-)$)

  **FOOO:** $L_1, L_2$ graded Lagrangian, $L_1 \cap L_2 = p$ of index 0

  $\sim L_1 \# L_2 \cong \text{Cone}(L_1 \rightarrow L_2)$

  $L_1[1]U_p L_2 \cong \text{Cone}(L_1 \rightarrow L_2)$

  $L_1 \# L_2$

  $L_2$

  $L_1$

- So e.g. consider $T^2$

  \[
  \begin{array}{c}
  \text{Cone}(\alpha \rightarrow \beta) \\
  \cong \text{disjointed Lagrangian} \\
  \cong \Sigma \oplus \Sigma
  \end{array}
  \]

  If we only started with $\alpha \oplus \beta$, derived would have $\Sigma \oplus \Sigma$
  but not $\Sigma_1 \oplus \Sigma_2$ separately; split closure address this.
If we start with 2 generators, successive Dehn twists give all homotopy classes of loops on $T^2$; but each homotopy class has only many non-Hamiltonian Lagrangians.

To generate $\mathcal{D}\text{Fuk}(T^2)$ as triangulated envelope we need e.g., 1 horizontal loop + only many vertical loops.

On the other hand, $\alpha$ and $\beta$ as above split generate key point: $\text{Cone}(\alpha \to \beta)$ gives directed sums of loops that may continuously within homotopy class.

But many cones & idealmats don't have an obvious geometric interpretation.

E.g. Clifford torus $T = \{(x,y,z)|x^2 + y^2 = 1\} \subset \mathbb{CP}^2$ has idealmats $\in H^*(T)$ without any obvious geometric interpretation.

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**Homological mirror symmetry conjecture** (Kontsevich 1994)

\[ X, X^\vee \text{ mirror Calabi-Yau} \Rightarrow D^b\text{Fuk}(X) \cong D^b\text{Co}(X^\vee) \]

and vice versa.

Let's see how this works at homology level in example of $T^2$ (after Polychachut - Zaslow).

Consider on symplectic side $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, \(\omega = \lambda dx \wedge dy (\int_{T^2} \omega = \lambda)\)

on complex side $X^\vee = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, $\tau = i\lambda$.

Lagrange submanifolds in $X$ can be isohed (Hamiltonianly) to straight lines with rational slope; likewise, we can arrange flat conftamilies to be constant.

We'll see: family of Lagrangians in homology class $(p,q)$ (+ $U(1)$ conn.) $\leftrightarrow$ family of rank $p$ vector bundles on $X^\vee$ with $c_1 = -q$. 

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We want to test:

\[ L_0 \leftrightarrow 0 \]

\[ L_1 \leftrightarrow \mathcal{L} \text{ degree } 1 \text{ line bundle} \]

\( L \leftrightarrow \Theta_p \text{ skyscraper sheaf of some point } p \)

Mirror to:

\[
\mathcal{L} \xrightarrow{\text{holom. sect.}} \mathcal{L}_{\text{ev}_p} \xrightarrow{\text{evaluation at } p} \Theta_p \text{ composition = value } s(p) \]

of \( s \) at \( p \) (in some trivialization of \( \mathcal{L}_{1p} \))

**First need to understand:** formulas for sections of holomorphic line bundles over the elliptic curve \( X^\vee \) ? \( \Rightarrow \) **Theta Functions**

- \( L \rightarrow X^\vee \) line bundle \( \rightarrow \) pullback to \( \text{mir. over } C \rightarrow X^\vee \) is holomorphically trivial; in fact, pullback to \( \mathbb{Z} \)-over \( C/\mathbb{Z} \) too.

\[
L = \mathbb{C} \times C/(z,v) \sim (z+1, v) \quad \text{where } \varphi \text{ holomorphic, } \varphi(z) \sim \varphi(z+1) = \varphi(z)
\]

\[
\varphi(z) = e^{-2\pi i z} e^{-\pi i z} \quad \text{determines degree } 1 \text{ line bundle } L
\]

with section the theta function \( \Theta(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2\pi i (\frac{z}{\tau} + nz)} \)

Generalization: \( \Theta[c', c''](\tau, z) = \sum_{m \in \mathbb{Z}} \exp(2\pi i (\tau \frac{(m+c')^2}{2} + (m+c')(z+c''))) \)

satisfies \( \Theta[c', c''](\tau, z+1) = e^{2\pi i c'} \Theta[c', c''](\tau, z) \)

\[
\Theta[c', c''](\tau, z + \tau) = e^{-\pi i z} e^{-2\pi i (z+c'')} \Theta[c', c''](\tau, z)
\]

\[
\tau \left( \frac{(m+c')^2}{2} + \tau (m+c') + (m+c')(z+c'') \right) = \tau \left( \frac{(m+1+c')^2}{2} - \frac{\tau}{2} + (m+1+c')(z+c'') - (z+c'') \right)
\]

- sections of \( L \otimes^n \) are \( \Theta[\frac{k}{n}, 0](n\tau, n\tau) \), \( k = 0 \ldots n-1 \)

- other line bundles are obtained by pullback by translation \( z \mapsto z+c'' \)
  ie. \( \Theta[0, c''] \) etc...