1 ✱ **Derived categories, Ext's & derived functors:**

1) • The der. cat. gives a better way to understand <u>derived functors</u>.

<u>Namely</u>: $F : \mathcal{A} \to \mathcal{B}$ left exact functor b/w abelian categories

$\mathcal{R} \subset \mathcal{A}$ is an adapted class of objects if

- $\mathcal{R}$ is stable under direct sums
- $C^\bullet$ acyclic complex in $\mathcal{R}$ $\Rightarrow$ $F(C^\bullet)$ acyclic
  $\hookrightarrow H^i(C) = 0$
- $\forall A \in \mathcal{A}$, $\exists$ inclusion $0 \to A \to R$, $R \in \mathcal{R}$. (<u>Ex</u>: injectives)

$K^+(\mathcal{R})$ = homotopy category of complexes bounded below of objects in $\mathcal{R}$
morphisms = chain maps up to homotopy

Then: $RF :=$ composition $D^+(\mathcal{A}) \xrightarrow[\text{by elts of } \mathcal{R}]{\text{resolution}} K^+(\mathcal{R}) \xrightarrow[F]{} D^+(\mathcal{B})$

SKIP

The functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is <u>exact</u>, ie. exact triangles $\mapsto$ exact triangles

Then $R^iF = H^i(RF)$ (What RF does for a single object $A \in \mathcal{A}$ is exactly what we do to compute $R^iF(A)$ using a resolution by objects of $\mathcal{R}$ & applying F, except taking cohomology).

2) Let $A, B \in \mathcal{A}$ (e.g. $Coh(X)$), view them as 1-step complexes in degree 0.

$B[k]$ shift ($B[k]^i = B^{i+k}$; so $B[k]$ concentrated in degree $-k$).

<u>Prop</u>: $\mathrm{Hom}_{D^b(\mathcal{A})}(A, B[k]) = \mathrm{Ext}^k_{\mathcal{A}}(A, B)$

✱ can use this to define product on $\mathrm{Ext}^k_{\mathcal{A}}(A,B) \otimes \mathrm{Ext}^\ell_{\mathcal{A}}(B,C) \to \mathrm{Ext}^{k+\ell}_{\mathcal{A}}(A,C)$ as composition in $D^b(\mathcal{A})$

<u>Example</u>: for $k=1$:

$$\begin{array}{ccccccc} 0 & \to & 0 & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & 0 & \to & 0 \end{array}$$

no chain maps; but we're allowed to invert quasi-isom's !!

If we have an extension

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ (s.e.s. in $\mathcal{A}$)

then we get maps of complexes

$$\begin{array}{ccccccc} 0 & \to & 0 & \to & C & \to & 0 \\ & & & & \uparrow g & & \\ 0 & \to & A & \xrightarrow{f} & B & \to & 0 \\ & & \downarrow id & & & & \\ 0 & \to & A & \to & 0 & \to & 0 \end{array}$$

quasi-isom.

②

which gives an element of $\mathrm{Hom}_{D^b(\mathcal{A})}(C, A[1])$ $\left(\simeq \mathrm{Ext}^1(C, A)\right)$
(can do the same with higher Ext's.)

* 2 ways to understand the proposition:
  - → if $\mathcal{A}$ has enough injectives, take an injective resol^n of B and replace B by quasi-isom. complex (not bounded, but $D^b \hookrightarrow D^+$ is full and faithful...)

    then chain maps
$$\begin{tikzcd}[column sep=small] & & & A \arrow[d] & & \\ I_0 \arrow[r] & \cdots \arrow[r] & I_{k-1} \arrow[r] & I_k \arrow[r] & I_{k+1} \arrow[r] & \cdots \end{tikzcd}$$
    up to homotopy $= H^k(\mathrm{Hom}(A, I_*))$.

  - → check definition of Ext as derived functor:

    say $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ s.e.s. in $\mathcal{A}$

    Then get an exact triangle in $D^b(\mathcal{A})$: $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$

    ($w =$ extension map as above)

    Axioms of triangulated categories ⇒

<u>Prop:</u>
$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[dl, "v"] \\ C \arrow[u, "{[1]}", "w"'] & \end{tikzcd}$$
exact triangle, E object ⇒ long exact sequences

$$\cdots \to \mathrm{Hom}(E, A[i]) \xrightarrow{u_*} \mathrm{Hom}(E, B[i]) \xrightarrow{v_*} \mathrm{Hom}(E, C[i]) \xrightarrow{w_*} \mathrm{Hom}(E, A[i+1]) \to \cdots$$
$$\cdots \to \mathrm{Hom}(A[i+1], E) \xrightarrow{w^*} \mathrm{Hom}(C[i], E) \xrightarrow{v^*} \mathrm{Hom}(B[i], E) \xrightarrow{u^*} \mathrm{Hom}(A[i], E) \to \cdots$$

applying to our case (A, B, C, E 1-step complexes) we get exactly the defining property of Ext as derived functor of Hom ✓.

(<u>Idea</u>: e.g., exactness at $\mathrm{Hom}(E, B)$: (same at other places)

- check $vu = 0$ for any exact triangle:
$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed, "\exists h"] & A[1] \arrow[d, "id"] \\ A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r] & A[1] \end{tikzcd}$$
axiom: $\exists h$ st. squares commute

h must be 0 ⇒ $vu = 0$ ✓

- now: assume $f: E \to B$ satisfies $vf = 0$.
$$\begin{tikzcd} E \arrow[r, "id"] \arrow[d, dashed, "\exists g"'] & E \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d, "0"] & E[1] \arrow[d, dashed] \\ A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & A[1] \end{tikzcd}$$
$\exists g$ st squares commute
⇒ $f = ug$

Hence $\ker v_* = \mathrm{Im}\, u_*$ ✓).

③ Last time we constructed the der. cat. of an abelian category, e.g. $D^b Coh(X)$

* <u>Derived Fukaya category</u>: similar story except $\mathcal{F}(M,\omega)$ is an A_∞-category, not an abelian category... construction proceeds via "twisted complexes"

1) additive enlargement $\Sigma\mathcal{A}$: allow formal direct sums and shifts of objects of $\mathcal{A}$

objects $:= \bigoplus_{\text{finite}} X_i[k_i]$ $\quad X_i \in \mathcal{A};\ k_i \in \mathbb{Z}$; $\quad hom\left(\bigoplus_i X_i[k_i], \bigoplus_j Y_j[l_j]\right) = \bigoplus_{i,j} hom(X_i, Y_j)[l_j - k_i]$

(write as matrices)

$$m_k(a_k, \dots, a_1)^{ij} = \sum_{i_1 \dots i_{k-1}} m_k(a_k^{i_{k-1} j}, \dots, a_1^{i i_1})$$

2) twisted complexes: $Tw\,\mathcal{A}$: objects = twisted complexes (X, δ_X)

$X = \bigoplus_i X_i[k_i] \in ob\ \Sigma\mathcal{A}$, $\quad \delta_X = (\delta_X^{ij}) \in hom^1_{\Sigma A}(X, X)$ degree 1 endo.

s.t.
$$\begin{cases} \bullet\ \delta_X \text{ strictly lower-triangular} \\ \bullet\ \sum_{k=1}^{\infty} m_k(\delta_X, \dots, \delta_X) = 0. \quad [\text{generalizes } \delta^2 = 0]. \end{cases}$$

↖ actually finite by strict triangular

Ex: $\begin{tikzcd} X_1[2] \arrow[r, "f"] \arrow[rr, bend right, "h"'] & X_2[1] \arrow[r, "g"] & X_3 \end{tikzcd}$

$g \in hom^0(X_2, X_3) = hom^1(X_2[1], X_3)$
$f \in hom^0(X_1, X_2)$
$h \in hom^{-1}(X_1, X_3)$

write $m_1 = \partial$, $m_2 = \cdot$: eqn become
$$\partial f = 0$$
$$\partial g = 0$$
$$g \cdot f = \partial h.$$

morphisms of twisted complexes: $hom_{Tw\,\mathcal{A}}((X, \delta_X), (Y, \delta_Y)) = hom_{\Sigma\mathcal{A}}(X, Y)$

but product structures deformed by differentials:

$$m_k^{Tw\mathcal{A}}(a_k, \dots, a_1) = \sum_{i_0 \dots i_k} m^{\Sigma\mathcal{A}}_{k+i_0+\dots+i_k}(\underbrace{\delta_{X_k} \dots \delta_{X_k}}_{i_k}, a_k, \underbrace{\delta_{X_{k-1}} \dots \delta_{X_{k-1}}}_{i_{k-1}}, \dots, a_1, \underbrace{\delta_{X_0} \dots \delta_{X_0}}_{i_0})$$

Ex: • $a \in hom(X, Y) \Rightarrow m_1^{Tw\mathcal{A}}(a) = m_1(a) + m_2(\delta_Y, a) + m_2(a, \delta_X)$
+ higher order terms

Think of this as generalizing diff! on prehom of complexes
(closed ⟷ "chain map", exact ⟷ "nullhomotopic")

• $m_2^{Tw\mathcal{A}}(a_2, a_1) = m_2(a_2, a_1)$ + higher order terms

④ $\mathrm{Tw}\,\mathcal{A}$ is a triangulated A_∞-category ($\exists$ mapping cones, like usual complexes)

3) cohomology category $D(\mathcal{A}) := H^0(\mathrm{Tw}\,\mathcal{A})$ (honest tri-cat.): same objects, but
$\hom(X,Y) := H^0(\hom^{\mathrm{Tw}\mathcal{A}}(X,Y),\ m_1^{\mathrm{Tw}\mathcal{A}})$ (NB: $\hom(X,Y[k]) = H^k(\ldots)$)
[analogue of: chain maps up to homotopy]
composition = induced by $m_2^{\mathrm{Tw}\mathcal{A}}$ on cohomology.

Rmk: there's no localization step wrt quasi-isoms:
"quasi-isomorphisms are built into the A_∞-structure and already invertible up to homotopy"?