① **Coherent sheaves on a complex mfld:**

$\mathcal{O}_X$ sheaf of holomorphic functions

$\rightsquigarrow$ a coherent sheaf $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules

(ie. U open set $\mapsto \mathcal{F}(U)$ module $/ \mathcal{O}_X(U)$
w/ nice properties wrt restrictions ...)

s.t. (1) $\mathcal{F}$ is of finite type (ie. $\exists$ open cover of X by U's s.t. $\mathcal{F}_{|U}$ is generated by a finite # of sections, ie. $\exists\ \mathcal{O}_{X|U}^{\oplus n} \twoheadrightarrow \mathcal{F}_{|U}$)

(2) $\forall \underset{\text{open}}{U \subset X}$, $\forall\ \phi: \mathcal{O}_{X|U}^{\oplus n} \to \mathcal{F}_{|U}$ hom. of $\mathcal{O}_X$-modules, $\ker(\varphi)$ is of finite type.

If X nice enough, $\Leftrightarrow$ $\mathcal{F}$ has <u>finite presentation</u> ie. $\exists$ open cover by subsets U st. $\exists$ exact seq. $\mathcal{O}_{X|U}^{\oplus m} \to \mathcal{O}_{X|U}^{\oplus n} \to \mathcal{F}_{|U} \to 0$.

ie. coherent sheaves are cokernels of morphisms of vector bundles

$\star$ Main advantages over bundles: kernels & cokernels of morphisms of coherent sheaves are coherent sheaves.

<u>Ex:</u> • E vector bundle $\Rightarrow$ E (loc. free) sheaf (of holom. sections)

• D hypersurface defined by $s=0$, s section of L line bundle

$$\Rightarrow \quad 0 \to L^{-1} \xrightarrow[s]{} \mathcal{O}_X \longrightarrow \mathcal{O}_D \to 0$$

(for small enough U, s.e.s. on sections over U ✓).

• more generally, $Z \subset X$ codim. r subvar. defined transversely as zero set of $s \in H^0(E)$ E rank r v.b.

$\Rightarrow$ Koszul resolution: $0 \to \Lambda^r E^* \xrightarrow[s]{} \Lambda^{r-1} E^* \xrightarrow[s]{} \cdots \to E^* \xrightarrow[s]{} \mathcal{O}_X \to \mathcal{O}_Z \to 0$

$\star$ X smooth $\Rightarrow$ coherent sheaves always have a finite resolution by vector bundles.

$\star$ <u>Ext groups:</u> = right derived functor of Hom.

<u>NB:</u> internal $\mathcal{H}om(\mathcal{E},\mathcal{F})$ = a sheaf

external $\mathrm{Hom}(\mathcal{E},\mathcal{F})$ = global sections of $\mathcal{H}om$ = a vector space.

In general: a functor $F: \mathcal{C} \to \mathcal{C}'$ is <u>left exact</u> if

$0 \to A \to B \to C \to 0$ short exact seq. $\Rightarrow$ $0 \to F(A) \to F(B) \to F(C)$

* Right derived functors: R^iF s.t.

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots$$

To compute, $R^iF(A)$, resolve A by injective objects (injective: $Hom(-,I)$ exact) ($\rightsquigarrow F$ become exact):

$$0 \to A \to I^0 \to I^1 \to I^2 \to \dots$$

then get a complex $0 \to F(I^0) \to F(I^1) \to F(I^2) \to \dots$

The cohomology of this complex gives $R^iF(A) = \dfrac{\ker(F(I^i) \to F(I^{i+1}))}{\mathrm{Im}(F(I^{i-1}) \to F(I^i))}$

($R^0F(A) = F(A)$ by left exactness)

Example: sheaf cohomology = right derived functor of global sections

Namely, ses of sheaves $\Rightarrow 0 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to \dots$

compute by resolving by acyclic sheaves (e.g. flasque sheaves ...)

(coincides with Čech cohomology)

* $Hom(\mathcal{E},-)$ (covariant) and $Hom(-,\mathcal{F})$ (contravariant) are left-exact

$Ext^i = R^i Hom$. In particular:

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \Rightarrow 0 \to Hom(\mathcal{E},\mathcal{F}_1) \to Hom(\mathcal{E},\mathcal{F}_2) \to Hom(\mathcal{E},\mathcal{F}_3) \to Ext^1(\mathcal{E},\mathcal{F}_1) \to Ext^1(\mathcal{E},\mathcal{F}_2) \to \dots$$

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0 \Rightarrow 0 \to Hom(\mathcal{E}_3,\mathcal{F}) \to Hom(\mathcal{E}_2,\mathcal{F}) \to Hom(\mathcal{E}_1,\mathcal{F}) \to Ext^1(\mathcal{E}_3,\mathcal{F}) \to Ext^1(\mathcal{E}_2,\mathcal{F}) \to \dots$$

In general, compute by resolving $\mathcal{F}$ by injectives (quasicoh., not coh.)

- since $Hom = H^0 \mathcal{H}om$, could try to first understand failure of exactness of $\mathcal{H}om$, then that of global sections.

Fact: | if $\mathcal{E}$ is locally free (i.e. vect bundle) then $\mathcal{H}om(\mathcal{E},-)$ is exact
Then $Ext^i(\mathcal{E},\mathcal{F}) = H^i(\mathcal{H}om(\mathcal{E},\mathcal{F}))$.

Otherwise, resolve $\mathcal{E}$ by locally free sheaves

$0 \to E_n \to \dots \to E_0 \to \mathcal{E} \to 0$ then we can build a complex of sheaves $\mathcal{H}om(E_n,\mathcal{F}) \leftarrow \mathcal{H}om(E_{n-1},\mathcal{F}) \leftarrow \dots \leftarrow \mathcal{H}om(E_0,\mathcal{F})$

whose hypercohomology computes $Ext^{\cdot}(\mathcal{E},\mathcal{F})$.

<u>Example:</u> $\mathcal{E}$ loc. free (vector bundle)
$\mathcal{O}_p$ skyscraper sheaf at a point

- $\mathcal{H}om(\mathcal{E},\mathcal{O}_p) = \mathcal{E}^* \otimes \mathcal{O}_p$ = skyscraper sheaf with stalk $\mathcal{E}^*_{|p}$ at p.

$$\rightarrow \begin{cases} Hom(\mathcal{E},\mathcal{O}_p) = \mathcal{E}^*_{|p} \\ Ext^i(\mathcal{E},\mathcal{O}_p) = 0 \quad \forall i \geq 1 \end{cases}$$ (skyscraper sheaves are acyclic)

- $\mathcal{H}om(\mathcal{O}_p,\mathcal{O}_p) = \mathcal{O}_p$ but this isn't the whole story...

Resolve $\mathcal{O}_p$ by locally free sheaves, e.g. use Koszul resolution

This is a local thing near p $\Rightarrow$ restricting, can assume X affine.

Then local coords. near p define a section s of $V \simeq \mathcal{O}_X^{\oplus n}$ ($n = \dim$)

$$0 \to \Lambda^n V^* \xrightarrow{s} \Lambda^{n-1} V^* \xrightarrow{s} \cdots \to V^* \xrightarrow{s} \mathcal{O}_X \xrightarrow{s} \mathcal{O}_p \to 0$$, apply $\mathcal{H}om(-,\mathcal{O}_p)$

gives $Ext^*(\mathcal{O}_p,\mathcal{O}_p)$ is hypercohomology of

$$\mathcal{O}_p \xrightarrow[0]{} V\otimes\mathcal{O}_p \xrightarrow[0]{} \cdots \to \Lambda^{n-1}V\otimes\mathcal{O}_p \xrightarrow[0]{} \Lambda^n V\otimes\mathcal{O}_p$$

i.e. since complex is trivial, $Ext^k(\mathcal{O}_p,\mathcal{O}_p) \simeq \Lambda^k V$.

- similarly, $Ext^*(\mathcal{O}_p,\mathcal{E})$ = hypercohomology of

$$\mathcal{E} \xrightarrow[s]{} V\otimes\mathcal{E} \xrightarrow[s]{} \cdots \xrightarrow[s]{} \Lambda^{n-1}V\otimes\mathcal{E} \xrightarrow[s]{} \Lambda^n V\otimes\mathcal{E}$$

can check this is exact except in last map, cokernel = skyscraper sheaf with stalk $(\Lambda^n V\otimes E)_{|p}$ at p. (in fact this is its Koszul resolution)

Hence $Ext^n(\mathcal{O}_p,\mathcal{E}) = \Lambda^n V \otimes E_{|p}$ ($\simeq E_{|p}$), all others zero.

Consistent with Serre duality: $Ext^i(\mathcal{E},\mathcal{F}) \cong Ext^{n-i}(\mathcal{F}, K_X\otimes\mathcal{E})^\vee$