Lecture 2 – Deformations of complex structures; Hodge theory

Recall: mirror symmetry predicts existence of mirror pairs of Calabi-Yau manifolds (= complex mfld with $\Omega^{n,0} \simeq \mathcal{O}_X$, ie. $\exists\, \Omega$ holom. vol. form)

$X, X^\vee$ mirror pair $\Rightarrow H^q(X, \Lambda^p T_X) \simeq H^q(X^\vee, \Omega^p_{X^\vee})$

+ isomorphism between "Yukawa couplings" on $H^{1,1}$ and on $H^1(X, TX)$.

On $H^{1,1}(X)$: $\langle \omega_1, \omega_2, \omega_3 \rangle := \int_X \omega_1 \wedge \omega_2 \wedge \omega_3$

$$+ \sum_{\substack{\beta \in H_2(X,\mathbb{Z}) \\ \beta \neq 0}} n_\beta \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \, \frac{e^{2\pi i \int_\beta \omega}}{1 - e^{2\pi i \int_\beta \omega}}$$

where n_β = "number of genus 0 complex curves in X representing the homology class β"

(defined in terms of Gromov-Witten invariants!)

$\triangle$ Z

On $H^1(X, T_X) \simeq H^{2,1}(X)$: $\langle \theta_1, \theta_2, \theta_3 \rangle := \int_X \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega)$

where $H^1(X, T_X)^{\otimes 3} \otimes H^0(X, \Omega^3_X) \longrightarrow H^3(X, \Lambda^3 T_X \otimes \Omega^3_X) = H^3(X, \mathcal{O}_X) = H^{0,3}(X)$

$\theta_1 \otimes \theta_2 \otimes \theta_3$ (0,1)-forms in T_X; Ω (3,0)-form; $\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega$ (3,3)-form w/ values in $\Lambda^3 TX$

or, in fact, thinking of θ_i as deformations of cx. structure J

$\rightarrow \Omega$ changes as we deform $J \rightsquigarrow \langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \Omega \wedge \nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega$

Need of course to assume Ω normalized for this to be ok.

Mirror symmetry prediction: if X & $X^\vee$ are mirror, then

$\langle \cdot,\cdot,\cdot \rangle$ on $H^{1,1}(X) \simeq H^1(X^\vee, TX^\vee)$, $\langle \cdot,\cdot,\cdot \rangle$

match under a certain change of coordinates, induced by mirror map $\mathcal{M}_{\text{Kähler}}(X) \rightarrow \mathcal{M}_{\text{complex}}(X^\vee)$ on tangent spaces

- Focus on: deformation of complex structures, coupling on $H^1(X, TX)$

②

<u>Reference:</u> Gross-Huybrechts-Joyce, "CY mflds & related geometries", ch. 14

- (X,J) almost complex $(J^2=-1)$ $\rightsquigarrow$ $TX\otimes\mathbb{C} = TX^{1,0}\oplus TX^{0,1}$

$$v^{1,0} = \tfrac{1}{2}(v - iJv), \quad v^{0,1} = \tfrac{1}{2}(v+iJv)$$

similarly, $\underset{}{T^*X\otimes\mathbb{C} \simeq \underset{\mathrm{span}(dz_i)}{T^*X^{1,0}} \oplus \underset{\mathrm{span}(d\bar z_i)}{T^*X^{0,1}}}$

$$\Lambda^k T^*X = \bigoplus_{p+q=k} \underset{\text{notation}}{\Lambda^{p,q}T^*X = \Omega^{p,q}(X)}$$

$(TX,J)\simeq TX^{1,0}$ <u>complex</u> vector bundle

- integrability of complex structure $\Leftrightarrow [T^{1,0},T^{1,0}]\subseteq T^{1,0}$

$\Leftrightarrow d = \partial+\bar\partial$ maps $\Omega^{p,q}\to\Omega^{p+1,q}\oplus\Omega^{p,q+1}$

$\Leftrightarrow \bar\partial^2 = 0$

then TX and assoc^d bundles are <u>holom.</u> vector bundles

Dolbeault cohomology: E holom. vect bundle $\Rightarrow$

$$C^\infty(X,E)\xrightarrow{\bar\partial}\Omega^{0,1}(X,E)\xrightarrow{\bar\partial}\Omega^{0,2}(X,E)\to \cdots$$

$$\rightsquigarrow H^q_{\bar\partial}(X,E) = \frac{\ker\bar\partial}{\operatorname{Im}\bar\partial}$$

- <u>Deforming</u> J to a nearby J':

$\Omega^{1,0}_{J'}\subseteq T^*X\otimes\mathbb{C} = \Omega^{1,0}_J\oplus\Omega^{0,1}_J$ is the graph of a linear map

$(-s):\Omega^{1,0}_J\to\Omega^{0,1}_J$. Conversely, recover J' from s: if s small enough

then $\Omega^{1,0}_{J'} := \mathrm{graph}(-s)$, $\Omega^{0,1}_{J'} = \overline{\Omega^{1,0}_{J'}}$ satisfy $T^*X\otimes\mathbb{C} = \Omega^{1,0}_{J'}\oplus\Omega^{0,1}_{J'}$

& set $J' = \begin{pmatrix} i & \\ & -i\end{pmatrix}$

Can also view s as section of $(\Omega^{1,0}_J)^*\otimes\Omega^{0,1}_J \simeq T^{1,0}_J\otimes\Omega^{0,1}_J$

ie. (0,1)-form with values in $T^{1,0}X$

$z_1,\dots,z_n$ local holom. coordinates for (X,J) $\Rightarrow$ $s = \sum s_{ij}\frac{\partial}{\partial z_i}\otimes d\bar z_j$

then basis of (1,0)-forms for J': $dz_i - s(dz_i) = dz_i - \sum_j s_{ij}\,d\bar z_j$

(0,1)-vector fields $\frac{\partial}{\partial\bar z_k} + s\left(\frac{\partial}{\partial \bar z_k}\right) = \frac{\partial}{\partial\bar z_k} + \sum_\ell s_{\ell k}\frac{\partial}{\partial z_\ell}$ pair trivially

③

- <u>Integrability?</u>

$\left(\bigoplus_q \Omega_X^{0,q} \otimes TX^{1,0}, \ \bar\partial\right)$ Dolbeault complex for $TX^{1,0}$ on (X,J)
($\bar\partial$ acts on forms only)

carries a Lie bracket $[\alpha\otimes v, \alpha'\otimes v'] := (\alpha\wedge\alpha')\otimes[v,v']$

$\rightsquigarrow$ diff'l graded Lie algebra (dgLa).

<u>Prop:</u> J' is integrable $\iff$ $\bar\partial s + \frac{1}{2}[s,s] = 0$

<u>Pf:</u> want: $\left[\frac{\partial}{\partial \bar z_i} + \sum_\ell s_{\ell i}\frac{\partial}{\partial z_\ell}, \ \frac{\partial}{\partial \bar z_j} + \sum_\ell s_{\ell j}\frac{\partial}{\partial z_\ell}\right] \in TX^{0,1}_{J'}$?

$$= \sum_\ell \left(\frac{\partial s_{\ell j}}{\partial \bar z_i}\frac{\partial}{\partial z_\ell} - \frac{\partial s_{\ell i}}{\partial \bar z_j}\frac{\partial}{\partial z_\ell}\right) + \sum_{k,\ell}\left(s_{ki}\frac{\partial s_{\ell j}}{\partial z_k}\frac{\partial}{\partial z_\ell} - s_{kj}\frac{\partial s_{\ell i}}{\partial z_k}\frac{\partial}{\partial z_\ell}\right)$$

$\in \operatorname{span}\left(\frac{\partial}{\partial z_\ell}\right)\ldots$

$\Rightarrow$ should be zero: <u>want:</u> $\forall i,j,\ell$,

$$\underbrace{\frac{\partial s_{\ell j}}{\partial \bar z_i} - \frac{\partial s_{\ell i}}{\partial \bar z_j}}_{\substack{\text{coeff}^t\text{ of } (d\bar z_i\wedge d\bar z_j)\otimes\frac{\partial}{\partial z_\ell} \\ \text{in } \bar\partial s}} + \underbrace{\sum_k\left(s_{ki}\frac{\partial s_{\ell j}}{\partial z_k} - s_{kj}\frac{\partial s_{\ell i}}{\partial z_k}\right)}_{\substack{\frac{1}{2}\cdot\text{coeff}^t\text{ of } d\bar z_i\wedge d\bar z_j\otimes\frac{\partial}{\partial z_\ell} \\ \text{in } [s,s]}} = 0$$

▲

- We'd like to understand $\mathcal{M}_{cx}(X) = \{J \text{ integrable cx. str. on } X\}/\mathrm{Diff}(X)$
or rather its <u>germ near X</u>

(<u>or:</u> assuming $\mathrm{Aut}(X,J)$ is discrete, near J $\exists$ universal family
$\mathfrak{X} \xrightarrow{\pi} \mathcal{U} \subset \mathcal{M}_{cx}$, $\mathfrak{X}, \mathcal{U}$ complex manifolds, π holomorphic,
fibers of π are $\simeq X$
any other family near J is induced by a classifying map & pullback from $\mathfrak{X}$.)

④ • $\{\text{integrable } J\text{'s}\} \underset{\text{locally}}{\simeq} \{ s \in \Omega^{0,1}(X, TX^{1,0}) \, / \, \bar\partial s + \frac{1}{2}[s,s] = 0 \}$

but need to quotient by Diff(X): $J \sim \phi^* J$

If ϕ is close to Id, $\phi^* J$ is close to $J \Rightarrow$ can be described as above.

$d\phi = \partial\phi + \bar\partial\phi$ where

$\partial\phi : TX^{1,0} \to \phi^* TX^{1,0}$ parts of $d\phi$ that commute / anticommute w/ J

$\bar\partial\phi : TX^{0,1} \to \phi^* TX^{1,0}$

$$\Rightarrow \phi^* dz_i = \underbrace{dz_i \circ \partial\phi}_{(1,0)} + \underbrace{dz_i \circ \bar\partial\phi}_{(0,1)} = \underbrace{\left(dz_i \circ \partial\phi\right)}_{(1,0)} \circ \left(Id + (\partial\phi)^{-1}\bar\partial\phi\right)$$

ie. $s = -(\partial\phi)^{-1}\bar\partial\phi$

• <u>Tangent space</u> – infinitesimal deformations ("over Spec $\mathbb{C}[t]/t^2$")

$J(t)$, $J(0) = J \rightsquigarrow s(t) \in \Omega^{0,1}(X, TX^{1,0})$, $\bar\partial s(t) + \frac{1}{2}[s(t), s(t)] = 0$

$\rightsquigarrow s_1 = \frac{ds}{dt}_{|t=0}$ satisfies $\bar\partial s_1 = 0$

Infinitesimal action of diffeomorphisms:

(ϕ_t), $\phi_0 = Id$, $\frac{d\phi}{dt}_{|t=0} = v$ vector field $\rightsquigarrow$

$$\frac{d}{dt}_{|t=0}\left(-(\partial\phi_t)^{-1}\bar\partial\phi_t\right) = -\frac{d}{dt}_{|t=0}\left(\bar\partial\phi_t\right) = -\bar\partial v$$

So: <u>first order deformations</u>

$$\mathrm{Def}_1(X,J) = \frac{\ker\left(\bar\partial : \Omega^{0,1}(X, TX^{1,0}) \to \Omega^{0,2}\right)}{\mathrm{Im}\left(\bar\partial : C^\infty(X, TX^{1,0}) \to \Omega^{0,1}\right)} = H^1(X, TX^{1,0})$$

In particular, given a family $\begin{matrix} \mathcal{X} \supset X \\ \downarrow \quad \downarrow \\ S \ni 0 \end{matrix}$ of deformations of (X, J) param^d by S

get a map $T_0 S \to H^1(X, TX)$ by looking at 1st order variations of J ....

<u>Kodaira-Spencer map</u>