Product structure: \[ \text{CF}^*(L_0, L_1) \otimes \text{CF}^*(L_1, L_2) \rightarrow \text{CF}^*(L_0, L_2) \]

Look at \( u: D^2 \rightarrow M \) J-holomorphic disk with:

- \( u(j) = p \in L_0 \cap L_1 \), \( u(j^2) = q \in L_1 \cap L_2 \), \( u(j^3) = r \in L_0 \cap L_2 \)
- \( u([j, j]) \subseteq L_0 \), \( u([j^2, j^3]) \subseteq L_1 \), \( u([j^3, j]) \subseteq L_2 \)

(or equivalently, \( u: \begin{array}{c} \text{Lo} \\ \text{L1} \\ \text{L2} \end{array} \rightarrow M \) Riem. surface of genus 0 with 3 ship-like ends [of finite energy])

Let \( M(p, q, r, [u], J) = \{ \text{such maps} \} \)

Expected dimension = \( \text{ind}([u]) = \deg r - (\deg p + \deg q) \)

(Where bicircle \( u^*TM \) & points graded lifts define the degrees)

Then set \( q \cdot p = \sum_{r \in L_0 \cap L_2} \frac{\# M(p, q, r, (\phi, J))}{\phi \in \pi_2, \text{ind}(\phi) = 0} \)

Notes:

- As usual, this is subject to achieving transversality, one-to-one...
- \( \text{Aut}(D^2) \) acts transitively on cyclically ordered triples of boundary points, so choice of \((j, j, j^2)\) is arbitrary.
- Lack of symmetry in \( \deg p, q, r \) of index formula is because the degree of \( r \in \text{CF}(L_0, L_2) \) is a minus that of \( r \in \text{CF}(L_2, L_0) \)

In general, we have a "Poincaré duality" \( \text{CF}^*(L, L') = \text{CF}^{-*}(L', L) \)

Compatible with differential, product, ...

Prop: If \( \text{ind}([u], \pi_2(M, L)) = 0 \) then the product satisfies Leibniz rule w.r.t.

\[ \text{differential, and hence induces a product on HF}^* \]

Moreover, the product on HF* is associative.

Idea pf: (i) for Leibniz rule: consider index 1 moduli...
compatibility by adding limit configurations: in the absence of bubbling, these are of 3 types:

\[ \partial (q \cdot p) \quad q \cdot (\partial p) \quad (\partial q) \cdot p \]

Gaining theorem: assuming transversality, adding these gives a 1-manifold with boundary.

\[ \# \text{ends} = 0 \quad (\text{w/ orientations, or mod 2}) \Rightarrow \text{Leibniz identity.} \]

\[ (\text{w/ signs depending on degree}) \]

Thus:

- \( p, q \) closed \( \Rightarrow \partial (q \cdot p) = \pm (\partial q) \cdot (\partial p) = 0 \)
- \( \partial p \) exact, \( q \) closed \( \Rightarrow q \cdot \partial p = \pm \partial (q \cdot p) \pm (\partial q) \cdot p \) exact.

\( \Rightarrow \) get product in \( HF^* \).

(2) associativity: we'll see now.

Higher operations:

\[ CF^*(L_0, L_1) \otimes \ldots \otimes CF^*(L_{k-1}, L_k) \xrightarrow{m_k} CF^*(L_0, L_k)[2-k] \]

Look at \( J \)-holomorphic maps

\[ D^2 \text{ with } (k+1) \text{ boundary marked pts} \]

(\text{Riem. surface w/ boundary, with } (k+1) \text{ ship-like ends})

\[ \exp \dim \mathcal{M}(p_1 \ldots p_k, q, [u], J) = \deg q - (\deg p_1 + \ldots + \deg p_k) + k-2 \]

The term \( k-2 \) comes from the \( \text{dim. of the moduli space of discs with} \]

\( k+1 \) \text{ marked points. Assume we can achieve transversality:}

Then \( m_k(p_1 \ldots p_k) := \sum_{q \in L_0 \cap L_k} \left( \# \mathcal{M}(p_1 \ldots p_k, q, [u], J) \right) T^{\text{cod} q} \]

\( [u] / \text{ind}=0 \)

\( (m_1 = \text{differential, } m_2 = \text{product}) \).
(3) Moduli space of discs with \((k+1)\) boundary marked points:

\[ M_{0,k+1} = \{(z_0,...,z_k) \in \mathbb{D}^2 \text{ distinct, in order}\} \text{ contractible, dim } k-2 \]

Complementary to moduli space \(\overline{M}_{0,k+1}\) of stable genus 0 Riemann surf. w/one \(\mathcal{D}\) component & \(k+1\) boundary marked pts, i.e. trees of discs attached together at marked nodal points, s.t. each component has \(\geq 3\) special points.

E.g: \(\overline{M}_{0,4}\) = closed interval

![Diagram of discs and nodes](image)

\[ \Rightarrow \text{when considering sequence of holom. discs as above, limit configurations allowed by Gromov compactness =} \]

- bubbling of spheres, of discs \(\{\text{energy accumulates at various places in domain}\}\)
- breaking of strips at marked pts
- degeneration of domain to \(\partial \overline{M}_{0,k+1}\)

Get relations when consider \(i\) of 1-dim! families of discs.

**Prop.** Assuming no bubbling of discs/spheres, we have \(\forall m \geq 1, \forall p_i \in L_i \cap L_i^*\),

\[ \sum_{k+l \geq 1} (-1)^* m_k(P_m, ..., P_{j+k+1}, m_k(P_{j+k+1}, ..., P_{j+1}), P_j, ..., P_1) = 0 \]

where \(* = \deg(p_1) + ... + \deg(p_j) + 1\)

\[ 0 \leq j \leq l-1 \]

**Ex.** \(m_1(m_1(p)) = 0; \ m_1(m_2(p,q)) + m_2(p, m_1(q)) + (-1)^{q+1} m_2(m_1(p), q) = 0 \)

Differential

\[ \text{Lehmiz rule} \]

next one: \(m_1(m_3(p, q, r)) \pm m_2(m_2(p, q), r) \pm m_2(p, m_2(q, r)) \pm m_3(m_1(p), q, r) \pm m_3(p, m_1(q), r) \pm m_3(p, q, m_1(r)) = 0 \)

**Says:** the product \(m_2\) is associative up to homotopy

(the homotopy being given by \(m_3\)).

& hence associative on cohomology.

... and so on.
Idea pf: consider a 1-dim moduli space \( \mathcal{M}(p_1,\ldots, p_m, q_1, \ldots, q_s) \) and its ends. Assuming transversality & absence of bubbling, limiting config are all of the form 
\[
\begin{array}{c}
\circlearrowleft_{P_1} & \circlearrowleft_{P_2} & \ldots & \circlearrowleft_{P_k} \\
q & P_{j+k} & P_m \\
\end{array}
\]
(there are the codim-1 strata, config with more components have higher codimension).

Total # ends \( = 0 \) = sum of items in the proposition
(Coeff of \( T^{w} (-)^{-l} q \) in \( \Sigma \ldots \) )

Def: \( Aoo-category \) = linear "category" where morphism spaces are equipped with such algebraic operations \( (m_k)_{k \geq 1} \).

Fukaya category = \( Aoo-cat \) with objects = Lagrangians
morphisms = Floer complexes
alg operations = as above.

(many versions with different details - we'll see later).

So far we have at best an \( Aoo \)-precategory i.e. morphisms and compositions are defined only for transverse objects.
(\( CF(L,L) = \ldots \))

* At the homology level, the Donaldson-Fukaya category (hom = HF) is easier to work with, but contains less information in general!