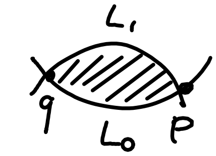
① $L_0, L_1 \subset (M,\omega)$ transverse Lagrangians $\to$ $CF(L_0,L_1) = \Lambda^{|L_0\cap L_1|}$

with differential $$\partial(p) = \sum_{\substack{q\in L_0\cap L_1 \\ \phi\in\pi_2 /\, \mathrm{ind}(\phi)=1}} \left(\#\mathcal{M}(p,q,\phi,J)/_{\mathbb{R}}\right) T^{\omega(\phi)}\, q$$

where $\mathcal{M} = \{$ finite energy J-hol. maps $u: \mathbb{R}\times[0,1]\to M$, $u(s,0)\in L_0$, $u(s,1)\in L_1$, $\lim_{s\to+\infty} u = p$, $\lim_{s\to-\infty} u = q$ $\}$



Limits of sequences in $\mathcal{M}$ have
- sphere bubbling (codim ≥ 2 if transv.)
- disc bubbling
- broken strips

} codim 1 if transv.

We've seen: if there is no bubbling (e.g.: if $\omega\cdot\pi_2(M,L_i)=0$) then $\partial^2=0$ (by considering ends of moduli spaces of index 2 strips).

---

* More about grading: want gradings on $CF(L_0,L_1)$ st. $\deg(q)-\deg(p)=$ index ?

Recall Maslov index $\Leftrightarrow$ $\pi_1(\Lambda Gr)=\mathbb{Z}$. Things are easier if $c_1(M)=0$ (or trivial on π_2)

then ΛGr-bundle of Lagr. planes over M admits a fiberwise universal cover — $\widetilde{\Lambda Gr}$-bundle of "graded Lagr. planes". Then, if at p we fix graded lifts of T_pL_i we can define the Maslov index of the intersection at p.

If L_1 is slightly clockwise from L_0,  $\mathbb{R}^n$, $(e^{-i\theta}\mathbb{R})^n$ then set $\deg(p)=0$

otherwise, set $\deg(p)=$ Maslov index from this reference configuration.

Obstruction to defining <u>globally</u> graded lift of L := <u>Maslov class</u> $\mu_L\in H^1(L,\mathbb{Z})$. If it vanishes then $\mathrm{ind}(u)=\deg(q)-\deg(p)$ depends only on p, q, not on the homotopy class $[u]$ $\Rightarrow$ Floer homology is $\mathbb{Z}$-graded.

Otherwise HF is only $\mathbb{Z}/N$-graded, $N=$ minimal Maslov number

... or can get $\mathbb{Z}$-graded theory by working over a larger ring, with an extra generator to keep track of Maslov index [in <u>monotone</u> case, where area & Maslov index are proportional, can just set $\deg(T)\neq 0$].

▲ <u>Note</u>: if L_i are oriented, then grading mod 2 $\equiv$ sign of intersection

② <u>Example:</u> $M = T^*N$, $\omega = \sum dp_i \wedge dq_i$

Equip N with a Riem. metric g, induces metric & a.c.s. on T^*N

(along zero section, $TM = TN \oplus T^*N$, $T^*N \simeq TN$ via g

Then $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$)

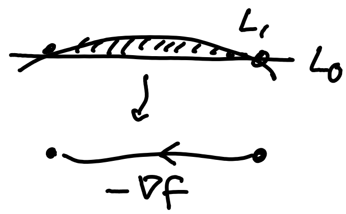
L_0 = zero section, L_1 = graph($\varepsilon\, df$), f Morse function on N, ε small

- $L_0 \cap L_1 = \mathrm{crit}(f)$

  Maslov index $\Leftrightarrow$ $n -$ Morse index of crit pt

Assume f is Morse-Smale (i.e. transversality for gradient flow lines):

- (Floer, Fukaya-Oh) For $\varepsilon \to 0$, holom. strips $\overset{1-1}{\longleftrightarrow}$ gradient flow trajectories



$\Rightarrow HF^*(L_0, L_1) \simeq HM_{n-*}(f)$ $(\simeq H^*(N))$

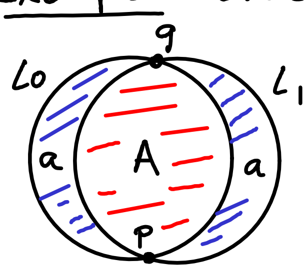
(can discard Novikov coeff's as all strips $p \to q$ have $\int u^*\omega = \varepsilon(f(q) - f(p))$)

---

* by Weinstein nbd thm, this is a univ local model for $L \subset M$ & a C^1-small Hamiltonian deformation of L. By Ham. isotopy inv^ce of HF, set $HF(L,L) := HF(L, \varphi(L))$. If L doesn't bound discs we conclude $HF(L,L) \cong H^*(L)$

If L does bound discs, but under a suitable assumption to ensure HF well-defined, e.g. L <u>monotone</u> ie. ω and Maslov positively proportional on $\pi_2(M, L)$, we have a filtration of Floer complex & a spectral sequence starting with $H^*(L; \Lambda)$ and converging to $HF(L,L)$ (with successive diff$\underline{l}$s = contributions of holom. discs of increasing area). $\rightarrow$ <u>Oh spectral sequence</u>.

<u>Example:</u> circle in $\mathbb{R}^2$: $H^*(L;\Lambda) = H^*(S^1, \Lambda) \Rightarrow HF(L,L) = 0$

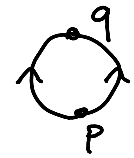


- $CF(L_0, L_1) = \Lambda p \oplus \Lambda q$, $\deg p = 0$, $\deg q = 1$.

  first page of Oh spectral seq: (thin strips):

  $\partial p = (T^a - T^a)\, q = 0$, $\partial q = 0 \mod T^A$

③

$\Rightarrow$ first page agrees with Morse complex for height function 

but second page includes strip 

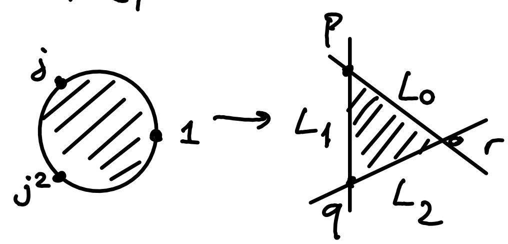
(index 1 strip from q to p, even though $\deg p - \deg q = -1$!
this is because converges to Maslov index $\underline{\underline{2}}$ disc  as $a \to 0$;
Floer homology is only $\mathbb{Z}/2$-graded)

Actually $\partial q = T^A p$ and $HF = 0$.

($\underline{\text{NB}}$: all other strips have at least one concave corner & hence cannot be rigid (index 1) strips).

$\underline{\text{Product structure}}$: $CF^*(L_0, L_1) \otimes CF^*(L_1, L_2) \to CF^*(L_0, L_2)$

Look at $u: D^2 \to M$ J-holom. disk with



$u(j) = p \in L_0 \cap L_1$, $u(j^2) = q \in L_1 \cap L_2$, $u(1) = r \in L_0 \cap L_2$

$u([1,j]) \subset L_0$, $u([j, j^2]) \subset L_1$, $u([j^2, 1]) \subset L_2$