Lagrangian Floer homology:

Recall: \((M,\omega)\) symplectic manifold \(\Rightarrow L_0, L_1\) compact Lagrangian submanifolds, \(L_0 \pitchfork L_1\) + J compat. a.e.s.

Neutral ring \(\Lambda = \{ \Sigma a_i \lambda_i / \lambda_i \to \infty \}\)

Floer complex \(CF(L_0, L_1) = \Lambda^{L_0 \pitchfork L_1}\) free \(\Lambda\)-module generated by \(L_0 \pitchfork L_1\).

Look at moduli space of J-holomorphic strips in a given class \(p \in \pi_2(M; L_0, L_1)\)

\[ u: \mathbb{R} \times [0, 1] \to M, \quad \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0 \]

- \(u(s, 0) \in L_0, u(s, 1) \in L_1\)
- \(u(s, t) \to p\) for \(s \to \pm \infty, q\) for \(s \to -\infty\)

Expected \(\dim \mathcal{M}(p, q, \phi, J) = \text{Maslov index} \)

"going from \(p\) to \(q\), signed # of times that \(T_{L_0} \& T_{L_1}\) become non-transverse"?

Want to define: \(\mathcal{F}(p) = \sum_{q \in L_0 \pitchfork L_1} \left( \# \mathcal{M}(p, q, \phi, J) \right) \tau(u(\phi)) \cdot \text{sympl. area} \)

\(\phi \in \pi_2/\text{ind}(\phi) = 1\)

Translation

Then (Floer):

If \(\{ \omega \}. \pi_2(M) = 0\) and \([\omega]. \pi_2(M, L_i) = 0\) then \(\mathcal{F}\) is well-defined, \(\mathcal{F}^2 = 0\), and HF is independent of chosen J & invariant under Hamiltonian deformations of \(L_0\) and/or \(L_1\).

Issues with the definition of \(\mathcal{F}\):

- Transversality is achieved for simple maps by picking generic J.
  For multiply covered maps (or configs with complicated bubbling), need various tricks... (domain-dependent J's, multi-valued perturbations, virtual cycles, ...)
- Orientation on moduli space: need auxiliary data, & top assumption on \(L_i\).
  Namely, if equip \(L_i\) with spin structure (i.e. double cover of frame bundle) then we get an orientation on moduli spaces.
- Compactness: again relies on Gromov's compactness theorem (for a fixed energy) there are now 3 types of phenomena.
2. Bubble of sphere: $|d\nu_n| \to \infty$ at interior point
   limiting configuration looks like

   Treatment similar to case of closed curves/GW invariants

   In good cases (if transversality can be achieved), config w/ sphere bubbles are a codim $\geq 2$ subset of the compactified $\overline{M}$

3. Bubble of disc: $|d\nu_n| \to \infty$ at boundary point
   limiting configuration looks like

   This is a severe technical issue because, in transverse case, bubbled configuration have real codim $1$, and contribute to $\mathfrak{D}\overline{M}$.

4. Breaking of ships: (or: energy escapes towards $s \to \infty$
   i.e. reparametrizing $u_n(-\delta_n, \cdot)$ have no limits)

   (or, if using domain $= \mathbb{D}^2 - \{\pm 1\}$, this is bubbling at $\pm 1$).

   limiting configuration looks like

   This is an in Nash theory, where a sequence of gradient flow lines can converge to a broken flow line.

* How to prove $\mathfrak{D}^2 = 0$ assuming no bubbling:

   consider $M(p,q,\phi,J)/_R$ for $\phi \in \pi_2$ of index 2

   $J$ generic

   This is expected to be a 1-dim moduli, which can be compactified by adding in broken trajectories

   $\overline{\left( M(p,q,\phi,J)/_R \right)} < \left( M(r,q,\phi_2,J)/_R \right)$

   (this is assuming no bubbling $\Rightarrow$ no other limiting scenarios!)

   Giving this $\Rightarrow$ the resulting $M(p,q,\phi,J)/_R$ is a manifold with boundary.

   Now: #ends of a compact 1-manifold, oriented (or counting mod 2), = 0.

   Ends = contributions to coefficient of $T^{\omega(\phi)}_q$ in $\mathfrak{D}^2(p)$.

* Bubble of disc is not just a technical issue to overcome, it's an actual obstruction to defining Floer homology.
Example: \( T^*S^1 \approx C_{F(L_0, L)} = \Lambda p \oplus \Lambda q \)

\[
\begin{align*}
\partial p &= \pm T \mathcal{A} \mathcal{U}(u) \quad q \\
\partial q &= \pm T \mathcal{A} \mathcal{U}(v) \quad p
\end{align*}
\]

so... \( \partial^2 \neq 0 \) ! what goes wrong: looks at moduli space of index 2 sheaves from \( p \) to itself. It's an interval...

(Parametrizing by upper half-disk \( \bigcirc \),

& setting \( \partial p L_1 \cup \text{unit circle}, \ L_0 = \text{real axis}, \)

there are: \( u_\alpha(z) = \frac{z^2 + \alpha}{1 + \alpha z^2} \), \( \alpha \in (-1, 1) \)

The two end points: 1) \( \bigcirc \) broken trajectory \( p \to q \to p \)

(contraining \( h \) \( \partial^2 p \))

2) \( \bigcirc \) contact ship at \( p \) \( + \) disc bubble with boundary in \( L_1 \)

So the disc bubble prevents \( \partial^2 = 0 \)...

* Hamiltonian isotopy ince: say \( H: [0,1] \times M \to \mathbb{R} \) generates \( \Phi^H_t = \text{flow of } X_H \) \( (t X_H \omega = dH) \)

Consider finite energy solution of
\[
\begin{align*}
\begin{cases}
\gamma: [-\varepsilon, \varepsilon] \to M \\
\frac{d\gamma}{ds} + \int \left( \frac{d\gamma}{dt} - \beta(s) X_H(t, u) \right) = 0 \\
u(s, 0) \in L_0, \ u(s, 1) \in L_1
\end{cases}
\end{align*}
\]

where \( \beta = \text{cutoff function} \)

\[
\begin{array}{c}
\text{CV to trajectory} \\
\delta(t) = X_H(t, s) \\
y(0) \in L_0, \ y(1) \in L_1 \\
\Leftrightarrow y(0) = q \in \Phi^H_{-\varepsilon}(L_0) \cap L_1
\end{array}
\]

( Set \( \bar{u}(s, t) = \Phi^H_{(t, 1)}(u(s, t)) \) then satisfies uncheked \( \overline{\partial}_y \cdot \bar{u}^2 = \text{for } s < 0) \)
Counting index 0 solutions gives \( \Psi_H : CF(L_0, L_1) \to CF(\Phi^*_H(L_0), L_1) \)
(isolated; no R-branched, invar now!)

In absence of disc bubbling, can show this is a chain map \( \Psi_H \circ \Phi^*_H = \Phi^*_H \circ \Psi_H \).
(idea: look at ends of index 1 moduli spaces = if no disc bubbling, they must be broken trajectories)

\[
\begin{array}{c}
\phi(L_0) \\
\Lambda_1
\end{array} \quad \Psi_H \quad \begin{array}{c}
\Phi^*_H(L_0) \\
\Lambda_1
\end{array}
\]

This chain map induces an isomorphism in homology.
(idea: look at \( \Psi_H \) and \( \Psi_{-H} \) for reversed isomopy, then build a homotopy between \( \Psi_H \circ \Psi_{-H} \) and \( \text{Id} \), etc.

...