① Lagrangian Floer homology:

Recall: (M,ω) symplectic manifold $\supset L_0, L_1$ compact Lagrangian submanifolds, $L_0 \pitchfork L_1$
+ J compat. a.c.s.

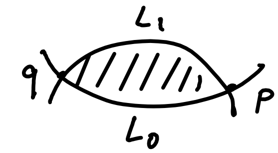
Novikov ring $\Lambda = \{ \sum a_i T^{\lambda_i} \,/\, \lambda_i \to +\infty \}$

Floer complex $CF(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ free Λ-module gen^{d} by $L_0 \cap L_1$.

Look at moduli space of J-holomorphic strips in a given class $\phi \in \pi_2(M; L_0, L_1)$

$$u: \mathbb{R}\times[0,1] \longrightarrow M, \quad \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$$

- $u(s,0) \in L_0$, $u(s,1) \in L_1$
- $u(s,t) \to p$ for $s \to +\infty$, q for $s \to -\infty$



expected $\dim_{\mathbb{R}} \mathcal{M}(p,q,\phi,J) =$ Maslov index

"= going from p to q, signed # of times that TL_0 & TL_1 become non-transverse"

- Want to define:
$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ \phi \in \pi_2 / \operatorname{ind}(\phi) = 1}} \left(\# \mathcal{M}(p,q,\phi,J)/_{\mathbb{R}} \right) T^{\omega(\phi)} q$$

$\omega(\phi)$ ← sympl. area

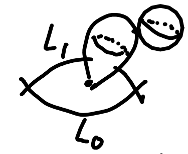
$/\mathbb{R}$ ← translation

Thm (Floer): if $[\omega]\cdot\pi_2(M) = 0$ and $[\omega]\cdot\pi_2(M, L_i) = 0$ then ∂ is well-def^{d}, $\partial^2 = 0$, and HF is indep^{t} of chosen J & invariant under Hamiltonian deformations of L_0 and/or L_1

Issues with the definition of ∂:

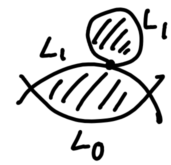
- transversality is achieved for simple maps by picking generic J.
  For multiply covered maps (or configs with complicated bubbling), need various tricks... (domain-dependent J's, multivalued perturbations, virtual cycles, ...)
- orientation on moduli space: need auxiliary data, & top. assumption on L_i.
  Namely: if equip L_i with spin structure (ie. double cover of frame bundle) then we get an orientation on moduli spaces.
- Compactness: again relies on Gromov's compactness theorem (for a fixed energy bound)
  there are now 3 types of phenomena.

② → <u>bubbling of spheres</u>: $|du_n| \to \infty$ at interior point
limiting configuration looks like



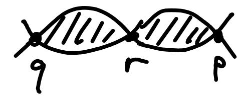
Treatment similar to case of closed curves / GW invariants
In good cases (if transversality can be achieved), configs. with sphere bubbles are a codim ≥ 2 subset of the compactified $\overline{\mathcal{M}}$

→ <u>bubbling of discs</u>: $|du_n| \to \infty$ at boundary point
limiting configuration looks like

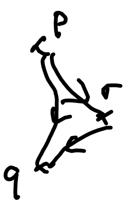


This is a serious technical issue because, in transverse case, bubbled configurations have real $\mathrm{codim}_{\mathbb{R}}\, 1$, and contribute to $\partial\overline{\mathcal{M}}$.

→ <u>breaking of strips</u>: (or: energy escapes towards $s \to \pm\infty$
i.e. reparametrizing $u_n(\cdot - \delta_n, \cdot)$ have $\neq$ limits)
(or, if using domain $= D^2 - \{\pm 1\}$, this is bubbling at ± 1).
Limiting configuration looks like



This is as in Morse theory, where a sequence of gradient flow lines can converge to a broken flow line



★ How to prove $\partial^2 = 0$ assuming no bubbling:

consider $\mathcal{M}(p, q, \phi, J)/\mathbb{R}$ for $\phi \in \pi_2$ of index 2, J generic

This is expected to be a 1-dim! mfld, which can be compactified by adding in broken trajectories $\displaystyle\coprod_{\substack{r \in L_0 \cap L_1 \\ \phi_1 \# \phi_2 = \phi}} \left(\mathcal{M}(p, r, \phi_1, J)/\mathbb{R}\right) \times \left(\mathcal{M}(r, q, \phi_2, J)/\mathbb{R}\right)$

(this is assuming no bubbling $\Rightarrow$ no other limiting scenarios!)

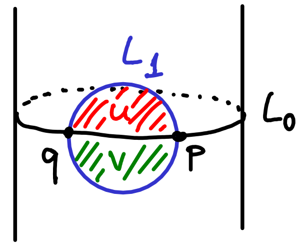
<u>Gluing thm</u> $\Rightarrow$ the resulting $\overline{\mathcal{M}(p, q, \phi, J)/\mathbb{R}}$ is a manifold w/ boundary.

Now: # ends of a compact 1-manifold, oriented (or counting mod 2), $= 0$.

Ends = contributions to coefficient of $T^{\omega(\phi)} q$ in $\partial^2(p)$.

★ Bubbling of discs is not just a technical issue to overcome, it's an actual obstruction to defining Floer homology

③

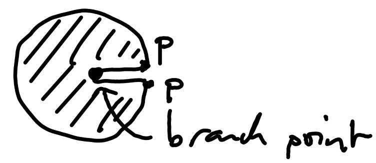
Example: $T^*S^1 \supset$ 

$CF(L_0, L_1) = \Lambda p \oplus \Lambda q$

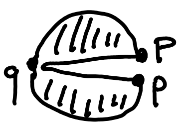
$\partial p = \pm T^{\text{area}(u)} q$

$\partial q = \pm T^{\text{area}(v)} p$

so... $\partial^2 \neq 0$! what goes wrong: look at moduli space of index 2 strips from p to itself. It's an interval...



(parametrizing by upper half-disc ,
& setting up L_1 = unit circle, L_0 = real axis,
these are: $u_\alpha(z) = \dfrac{z^2+\alpha}{1+\alpha z^2}$, $\alpha \in (-1,1)$)

The two end points: 1) 

broken trajectory $p \to q \to p$
(contributing to $\partial^2 p$)

2) 

constant strip at p
+ disc bubble with boundary in L_1

So the disc bubble prevents $\partial^2 = 0$ ...

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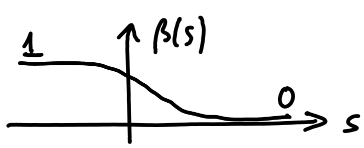
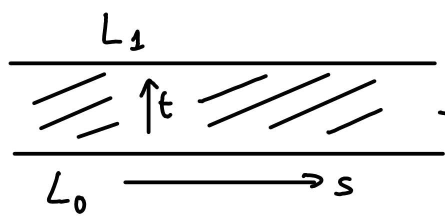
* Hamiltonian isotopy inv^{ce}: say $H: [0,1] \times M \to \mathbb{R}$ generates ϕ_H^t = flow of X_H ($\iota_{X_H}\omega = dH$)

Consider finite energy solutions of

$$\begin{cases} u: \mathbb{R} \times [0,1] \longrightarrow M \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - \beta(s) X_H(t,u)\right) = 0 \\ u(s,0) \in L_0, \quad u(s,1) \in L_1 \end{cases}$$

where β = cutoff function 

cv to trajectory
$\dot{\gamma}(t) = X_H(t,\gamma)$
$\gamma(0) \in L_0$, $\gamma(1) \in L_1$
$\Leftrightarrow \gamma(1) = q \in \phi_H^1(L_0) \cap L_1$

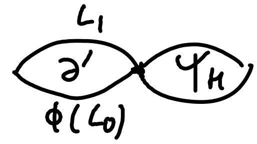
$\leftarrow$ 

$\to$ cv to $p \in L_0 \cap L_1$

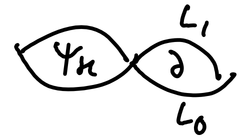
(set $\tilde{u}(s,t) = \phi_H^{(t,1)}(u(s,t))$ then satisfies unperturbed $\bar{\partial}_J$-eq^{n} for $s \ll 0$).

(4) Counting <u>index 0</u> solutions gives $\Psi_H : CF(L_0, L_1) \to CF(\phi_H^1(L_0), L_1)$
(isolated; no $\mathbb{R}$-transl. inv^{ce} now!)

In absence of disc bubbling, can show this is a <u>chain map</u> $\Psi_H \circ \partial = \partial' \circ \Psi_H$.
(<u>idea</u>: look at ends of index 1 moduli spaces = if no disc bubbling, they must be broken trajectories



or



)

+ this chain map induces an isomorphism in homology.
(idea: look at Ψ_H and Ψ_{-H} for reversed isotopy, then build a homotopy between $\Psi_H \circ \Psi_{-H}$ and Id ....).