Lagrangian Floer homology:

\[(M, \omega) \text{ symplectic manifold } \Rightarrow L_0, L_1 \text{ compact Lagrangian submanifolds}\]

**Formally**, Floer homology = Morse theory for "action functional" on path space \(P(L_0, L_1)\), where crit pts are constant paths & gradient flows = J-hol. ships.

More precisely: \(\forall \gamma \in \Gamma(L_0, L_1)\) (path from \(L_0\) to \(L_1\)):

\[
dA(\gamma) : V = \int_{[0,1]} \nabla(\gamma, v) dt = \int_{[0,1]} \varphi(\gamma, v) dt = \langle J\gamma, v \rangle_{L^2}
\]

where \(\gamma(t)\) is a path from \(L_0\) to \(L_1\), and \(v(0) \in T_{\gamma(0)}L_0\), \(v(1) \in T_{\gamma(1)}L_1\).

**Hence**: crit pts = const. paths \(\dot{\gamma} = 0\); gradient flows = J-hol. maps \(\frac{d\gamma}{ds} = -J\dot{\gamma}\)

**Difficult**: to define rigorously as \(\dim M\) Morse theory, so use holom. curves instead.

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**Actual setup**: Assume \(L_0 \cap L_1 = L_0 \cap L_1\) finite set.

Recall Novikov ring \(\Lambda = \{ \sum_{k=1}^{\infty} \lambda_k \tau \lambda_k / \lambda_k \to \infty \}\)

Floer complex \(CF(L_0, L_1) = \Lambda^{L_0 \cap L_1}\) free \(\Lambda\)-module gen'd by \(L_0 \cap L_1\).

**Goal**: define a differential \(\partial\) by counting holomorphic discs:

Look at:\(u : \mathbb{R} \times [0,1] \to M\) equipped with \(J\) \(-\)-comp. a.c.s.

\(s.t.
\begin{align*}
\begin{cases}
\nabla_j u = 0, & \text{i.e. } \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0 \\
u(s,0) \in L_0, \quad u(s,1) \in L_1 \\
\lim_{s \to +\infty} u(s,t) = p, \quad \lim_{s \to -\infty} u(s,t) = q \\
\text{the energy } E(u) = \int_u \omega = \iint \left| \frac{\partial u}{\partial s} \right|^2 < \infty
\end{cases}
\end{align*}
\)

\[(NB: \mathbb{R} \times [0,1] \sim \mathbb{D}^2 \setminus \{ \pm 1 \} \text{ so also think of maps } \gamma : \mathbb{D} \to M)\]

\(M(p, q, [\nu], J) = \{ u \text{ solns of } (\gamma) \}\) nodal space of holom discs.

\(\pi_2(M; L_0, L_1)\) homotopy class

\((\gamma)\) is a Fredholm problem, \(\text{exp. dim. } M = \text{ind}(\nabla_j)\)

\(\text{ind}(\nabla_j) = \text{Maslov index}\)

come from \(\pi_4(\Lambda Gr) = \mathbb{Z}\) for Lagrangian Grassmannian in \(\mathbb{R}^{2n}\).
Let $L_0, L_1(t), t \in [0,1]$ be Lagrangian subspaces of $\mathbb{R}^{2n}$, s.t. $L_0(0), L_1(1) \cap L_0$.

Then Maslov index of the path $L_1(t) := \#$ times that $L_1(t)$ fails to be transverse to $L_0$ (counted with sign & multiplicity).

Example: path $(e^{it}R)_{t=0}^{T} \times (e^{i\theta_0}R)$ if $\theta_0$ through 0, then $\mu(L_0, L_1(t)) = n$.

Now: given a strip $u$, trivialize $u^*TM \to u^*TL_0, TL_1$ paths of Lagrangians. Can trivialize so that $TL_0$ remain constant.

Then $\text{ind}(u) := \text{Maslov index of path } TL_1 \text{ relative to } TL_0$ as one goes from $p$ to $q$.

Example: $q \to p \to \in [\mathbb{R}^2]$ has $\text{ind}(u) = 1$.

Synth area $\phi \in \phi_\Sigma$.

Want to define: $\Theta(p) = \sum_{q \in L_0 \cap L_1, \phi \in \pi_2/\text{ind}(\phi) = 1} (\# M(p, q, \phi, J)/R) \cdot \Sigma u(\phi)$

Issues:
- transversality
- compactness, bubbling
- orientation of $M$ (signed counts? triple work over $\mathbb{Z}/2$)
- $\Sigma^2 = 0$?

A special case when bubbling isn't an issue:

Then (Floer):

If $[w].\pi_2(M) = 0$ and $[w].\pi_2(M, L_i) = 0$

then $\Theta$ is well-defined, $\Sigma^2 = 0$, and HF is indep of chosen $J$ & invariant under Hamiltonian deformations of $L_0$ and/or $L_1$.

Corollary:

$[w].\pi_2(M, L) = 0$, $\psi$ Ham. diffeo, $\psi(L) \cap L \Rightarrow |\psi(L)\cap L| \geq \Sigma \text{bi}(L)$

(Special case of Arnol'd conjecture; idea: $HF(L, \psi(L)) \simeq H^*(L)$; rank $CF \geq \text{rank} HF$)

Example: $T^*S^1 = \mathbb{R} \times S^1$

$\text{CF}(L_0, L_1) = \Lambda_p \otimes \Lambda_q$

$\Theta_p = (\text{Tarea}(w) - \text{Tarea}(v)) q$

$\Theta_q = 0$. 

In this case $\exists$ well-def $\varphi$-grading (Riesz index only depends on $p, q$, not on homotopy class of $\varphi$), e.g. $\deg(p) = 0, \deg(q) = 1$.

2 cases:
- $\text{area}(u) = \text{area}(v)$ ($\mathcal{L}_0, \mathcal{L}_1$, Ham. isohoric): $HF(\mathcal{L}_0, \mathcal{L}_1) = \pi_1^*(\mathbb{S}^1, \wedge)$
- $\text{area}(u) \neq \text{area}(v)$ ($\mathcal{L}_0, \mathcal{L}_1$, can be disjointed): $HF(\mathcal{L}_0, \mathcal{L}_1) = 0$.

Back to issues with the definition of $\mathcal{D}$:
- Transversality is achieved for simple maps by picking generic $\varphi$.
- For multiply covered maps (or conigs with complicated bubbling), need various tricks... (domain-dependent $\varphi$'s, multivalued perturbations, virtual cycles, ...)
- Orientation on moduli space: need auxiliary data, & top assumption on $\mathcal{L}_i$.
  - Namely: if equip $\mathcal{L}_i$ with spin structure (i.e. double cover of frame bundle) then we get an orientation on moduli space.