(1) <u>Recall:</u> calculating periods of $\check{\Omega}_\psi$ on $\check{X}_\psi = \widetilde{X_\psi/G}$ mirror quintic family

- we calculated <u>one</u> period by hand: $\int_{\check{T}_0} \check{\Omega}_\psi$ proportional to
$$\underline{\phi_0(z) = \sum_{n\geq 0} \frac{(5n)!}{(n!)^5} z^n}, \quad \text{where } z = (5\psi)^{-5}. \quad (z \to 0 \text{ LCSL}).$$
- we showed that <u>all</u> periods of $\check{\Omega}_\psi$ satisfy Picard-Fuchs equation
$$(\star)\ \mathcal{D}\phi = 0, \text{ where } \mathcal{D} = \Theta^4 - 5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4) = F(\Theta)$$
$$\left(\Theta = z\frac{d}{dz}\right)$$
- We've seen: can rewrite $\mathcal{D}\phi = 0$ as a 1^{st} order system
for $\vec{w} = \begin{pmatrix} \phi \\ \Theta\phi \\ \Theta^2\phi \\ \Theta^3\phi \end{pmatrix}$, $\Theta\vec{w} = A(z)\vec{w}$, $A(z)$ holomorphic, $A(0) =: N = \begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & 0 \end{pmatrix}$

Fundamental solution is of the form $\Phi(z) = \underbrace{S(z)}_{\text{holomorphic}} \exp(N \log z)$

First row $\Phi(z)_{1i}$ $1\leq i \leq 4$ = fundamental system of solutions for $(\star)$
- Monodromy: z around 0 gives $\Phi(z) \mapsto \Phi(z)\exp(2\pi i N)$
so first column is monodromy-invariant, second changes by $2\pi i \cdot$(first), ...

<u>Relevance:</u> if $\omega(z) = \int_\beta \Omega$ is a period then it's a sol$^{\underline{n}}$ to Picard-Fuchs
$\Rightarrow$ it's a linear combination of fund$^{\underline{l}}$ solutions
= first row of matrix $\Phi(z)$

so $\exists$ basis $\alpha_1 \ldots \alpha_4$ of $H_3(\check{X}, \mathbb{C})$ s.t. $\int_{\alpha_i} \Omega = \Phi(z)_{1i}$

The monodromy transformation in this basis is then $T = e^{2\pi i N}$ (max. unipotent)

- <u>More periods of Ω:</u> we already have a sol$^{\underline{n}}$ $\phi_0(z)$ which is analytic, single-valued. By above, it's the only one up to scaling.
Next we'd like a multivalued solution $\phi_1(z)$ s.t.
$$\phi_1(ze^{2\pi i}) = \phi_1(z) + 2\pi i\, \phi_0(z)$$
($\leftrightarrow$ desired behavior for next fundamental sol$^{\underline{n}}$!
& up to scaling, for period of Ω on β_1 s.t. $\beta_1 \xmapsto{\text{monodromy}} \beta_1 + \beta_0$).

②

Recall: $\underline{\phi_1(z) = \phi_0(z)\log z + \tilde\phi(z)}$, $\tilde\phi$ holomorphic

Let's find $\tilde\phi$. First note: $\Theta^i\big(f(z)\log z\big) = \big(\Theta^i f(z)\big)\log z + i\,\Theta^{i-1} f(z)$

(because $\Theta = z\frac{\partial}{\partial z} = \frac{\partial}{\partial \log z}$; product rule or induction)

so: if we write $F(x) = x^4 - 5z(5x+1)\cdots(5x+4)$, then

$$\mathcal{D}\phi_1(z) = F(\Theta)\big(\phi_0(z)\log z + \tilde\phi(z)\big)$$
$$= \underbrace{\big(\mathcal{D}\phi_0(z)\big)}_{=0}\log z + F'(\Theta)\,\phi_0(z) + \mathcal{D}\tilde\phi(z)$$

$\Rightarrow \underset{4^{\text{th}}\text{ order}}{\mathcal{D}}\tilde\phi(z) = -F'(\Theta)\,\phi_0(z)$ gives a recurrence relation on the Taylor coefficients of $\tilde\phi$

calculate explicitly ... $\rightarrow$ $\underline{\tilde\phi(z) = 5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)z^n}$

- Now, $\underline{\text{canonical coordinate}}$: recall $\beta_0, \beta_1 \in H_3(\check{X}_4, \mathbb{Z})$, $\underset{\text{monodromy}}{\beta_1 \mapsto \beta_1 + \beta_0}$

Then $\int_{\beta_0}\check\Omega = C\,\phi_0(z)$

while $\int_{\beta_1}\check\Omega = C'\phi_0(z) + C''\phi_1(z)$

monodromy acts: $C'\phi_0 + C''\phi_1 \mapsto C'\phi_0 + C''(\phi_1 + 2\pi i\,\phi_0)$

want $\int_{\beta_1}\check\Omega \mapsto \int_{\beta_1+\beta_0}\check\Omega$ $\Rightarrow 2\pi i\,C'' = C$.

Then canon. coords: $w = \dfrac{\int_{\beta_1}\check\Omega}{\int_{\beta_0}\check\Omega} = \dfrac{C'}{C} + \dfrac{1}{2\pi i}\dfrac{\phi_1}{\phi_0}$

$$= \frac{1}{2\pi i}\log c_2 + \frac{1}{2\pi i}\log z + \frac{1}{2\pi i}\frac{\tilde\phi(z)}{\phi_0(z)}$$

$$\underline{q = \exp(2\pi i w) = c_2\, z\, \exp\left(\frac{\tilde\phi(z)}{\phi_0(z)}\right)}$$

× constant because don't know β_1 exactly, only up to adding a multiple of β_0

can write power series ....

③

- <u>Yukawa coupling</u> on $H^{2,1}(\check{X})$:

let $W_k = \int_{\check{X}_z} \check{\Omega}(z) \wedge \frac{d^k}{dz^k} \check{\Omega}(z)$ (still same family of $\check{\Omega}$!).

Rewrite Picard Fuchs in form $\frac{d^4}{dz^4}[\check{\Omega}] + \sum_{k=0}^{3} C_k(z) \frac{d^k}{dz^k}[\check{\Omega}] = 0$

$\rightsquigarrow$ then $W_4 + \sum_{k=0}^{3} C_k W_k = 0$

But Griffiths transversality $\Rightarrow$ $\check{\Omega}$ & 1^{st}, 2^{nd} derivatives have no $(0,3)$ component $\Rightarrow$ $W_0 = W_1 = W_2 = 0$.

Moreover:

$$0 = \frac{d^2 W_2}{dz^2} = \int \frac{d^2 \check{\Omega}}{dz^2} \wedge \frac{d^2 \check{\Omega}}{dz^2} + 2 \int \frac{d\check{\Omega}}{dz} \wedge \frac{d^3 \check{\Omega}}{dz^3} + \int \check{\Omega} \wedge \frac{d^4 \check{\Omega}}{dz^4}$$

$$= 0 + 2\left(\frac{dW_3}{dz} - W_4\right) + W_4$$

so $W_4 = 2W_3'$, and get $W_3' + \frac{1}{2} C_3 W_3 = 0$!

Look at coeff^t of $\frac{d^3}{dz^3}$ in Picard-Fuchs $\rightsquigarrow C_3(z) = \frac{6}{z} - \frac{2 \cdot 5^5}{1-5^5 z}$

$\Rightarrow (\log W_3)' = -\frac{3}{z} + \frac{5^5}{1-5^5 z}$. Integrating we get

$W_3(z) = \frac{c_1}{(2\pi i)^3 z^3 (5^5 z - 1)}$. This is almost $\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle$.

Still need to <u>normalize</u> (want $\langle \dots \rangle$ rel. to $\frac{\check{\Omega}}{\int_{\beta_0} \check{\Omega}}$ not $\check{\Omega}$)

and switch to <u>canonical coordinates</u> (want $\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle$ not $\frac{\partial}{\partial z}$).

- <u>normalization</u>: scaling $\check{\Omega}$ by $f(z)$ changes $\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \int_X \check{\Omega} \wedge \frac{\partial^3}{\partial z^3} \check{\Omega}$ by $f(z)^2$
(no derivatives of f come up because $\check{\Omega} \wedge \frac{\partial^i}{\partial z^i} \check{\Omega} \equiv 0$ for $i<3$)
In our case, want to scale by $\frac{1}{\int_{\beta_0} \check{\Omega}} = \frac{\text{const}}{\phi_0(z)}$

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$$\Rightarrow \left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right\rangle = \frac{c_1}{(2\pi i)^3 z^3 (5^5 z - 1)\, \phi_0(z)^2}$$

- <u>switching coordinates</u>: $\frac{\partial}{\partial w} = \left(\frac{dw}{dz}\right)^{-1} \frac{\partial}{\partial z} \Rightarrow \left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right\rangle = \frac{c_1}{(5^5 z - 1)\phi_0(z)^2 \delta(z)^3}$

where $\delta(z) = 2\pi i\, z \frac{dw}{dz} = z \frac{d \log q}{dz} = 1 + z \frac{d}{dz}\left(\frac{\tilde{\phi}(z)}{\phi_0(z)}\right)$

Finally we want to expand this as a power series in q.
Since $dq/dz = q \frac{d\log q}{dz} = \frac{q}{z} \delta(z) = c_2\, \delta(z) \exp\left(\tilde{\phi}/\phi_0\right)$, we have

$$\frac{d^j}{dq^j}\left(\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right\rangle\right) = \left(\frac{1}{c_2 \delta(z) \exp(\tilde{\phi}/\phi_0)} \frac{d}{dz}\right)^j \left(\frac{c_1}{(5^5 z - 1)\phi_0(z)^2 \delta(z)^3}\right)$$

$\rightarrow$ calculate expansion in q by evaluating these at $z = 0$ (from expansion of $\phi_0(z), \tilde{\phi}(z)$)

get: $\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right\rangle = -c_1 - 575 \frac{c_1}{c_2} q - \frac{1950750}{2} \frac{c_1}{c_2^2} q^2 - \frac{10277490000}{6} \frac{c_1}{c_2^3} q^3$
$- \frac{74486048625000}{24} \frac{c_1}{c_2^4} q^4 + \dots$

These coefficients $\leftrightarrow$ Gromov-Witten invts of quintic