Recall calculating periods of $\dot{X}_\psi$ on $\dot{X}_\psi = X_\psi / \mathcal{G}$ mirror quintic family

- We calculated one period by hand: \( s_\psi \dot{X}_\psi \) proportional to \[
\phi_0(z) = \sum_{n=0}^{\infty} \frac{(5^n)}{n!} z^n, \quad \text{where} \quad z = (5\psi)^{-1/5}, \quad (z \to 0 \text{ LCSL}).
\]

- We showed that all periods of $\dot{X}_\psi$ satisfy Picard-Fuchs equation

\[\Phi(z) = 0, \quad \text{where} \quad \Phi(z) = z^4 - 5 z^3 (5\psi + 1)(5\psi + 2)(5\psi + 3)(5\psi + 4) - \Phi(z) \quad (\Phi(z) = z \frac{d}{dz})
\]

- We've seen, can rewrite $\Phi(z) = 0$ as a 1st order system

\[\dot{w} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}, \quad \text{where} \quad \Phi(z) = \begin{pmatrix} A(z) \\ A(z) \hat{A}(z) \end{pmatrix}, \quad \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}
\]

Fundamental solution is of the form \( \Phi(z) = S(z) \exp(N \log z) \)

- First row $\Phi(z)_{1i}$, $1 \leq i \leq 4$ = fundamental system of solutions for (a)

- Monodromy: $z$ around 0 gives $\Phi(z) \to \Phi(z) \exp(2\pi i N)$

so first column is monodromy-invariant, second changes by $2\pi i \text{ (first)}$...

Relevance: if $\omega(z) = \sum_b s_b$ is a period, then it's a soln. to Picard-Fuchs

\[ \Rightarrow \text{it's a linear combination of fund. solutions} \]

\[ = \text{first row of matrix } \Phi(z) \]

\[ \exists \text{ basis } \alpha_1, \ldots, \alpha_4 \text{ of } H_3(X, \mathbb{C}) \text{ s.t. } \sum_b \omega_b = \Phi(z)_{1i} \]

The monodromy transformation in this basis is then $T = e^{2\pi i N}$ (max. unipotent)

- More periods of $\omega$: we already have a soln. $\phi_0(z)$ which is analytic, single-valued. By above, it's the only one up to scaling.

Next we'd like a multivalued solution $\phi_1(z)$ s.t.

\[ \phi_1(z e^{2\pi i}) = \phi_1(z) + 2\pi i \phi_0(z) \]

(\( \leftrightarrow \) desired behavior for next fundamental soln.)

& up to scaling, for period of $\omega$ on $\beta$, s.t. $\beta \to \beta + \beta_0$.
Necessary: \( \phi_1(z) = \phi_0(z) \log z + \bar{\phi}(z) \), \( \bar{\phi} \) holomorphic

Let's find \( \bar{\phi} \). First note: \( \theta' (f(z) \log z) = (\theta' f(z)) \log z + i \theta' f(z) \)

(because \( \theta = z \frac{2}{dz} = \frac{2}{\log z} \); product rule and \( \log z \))

so if we write \( F(z) = z^4 - 5z(5z+1)(5z+4) \), then

\[
\begin{align*}
\nabla \phi_1(z) &= F(\theta) (\phi_0(z) \log z + \bar{\phi}(z)) \\
&= (\nabla \phi_0(z)) \log z + F'(\theta) \phi_0(z) + \nabla \bar{\phi}(z)
\end{align*}
\]

\( \Rightarrow \nabla \bar{\phi}(z) = -F'(\theta) \phi_0(z) \) gives a recurrence relation on the Taylor coefficients of \( \bar{\phi} \)

\( \theta \)-order

Calculate explicitly ... \( \bar{\phi}(z) = 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(5n)^5} \left( \frac{5n}{j=n+1} \frac{1}{j} \right) z^n \)

* Now, canonical coordinates: recall \( \beta_0, \beta_1 \in H_3(\mathbb{C}, \mathbb{C}) \), \( \beta_1 \mapsto \beta_1 + \beta_0 \)

Then \( \int_{\beta_0} \bar{\phi} = C \phi_0(z) \)

while \( \int_{\beta_1} \bar{\phi} = C' \phi_0(z) + C'' \phi_1(z) \)

random acts: \( C' \phi_0 + C'' \phi_1 \mapsto C' \phi_0 + C'' (\phi_1 + 2\pi i \phi_0) \)

and \( \int_{\beta_1} \bar{\phi} \mapsto \int_{\beta_1 + \beta_0} \bar{\phi} \Rightarrow 2\pi i C'' = C \).

Then canonical cords: \( \psi = \frac{\int_{\beta_1} \bar{\phi}}{\int_{\beta_0} \bar{\phi}} = \frac{C'}{C} + \frac{1}{2\pi i} \frac{\phi_1}{\phi_0} \)

\[ = \frac{1}{2\pi i} \log c_2 + \frac{1}{2\pi i} \log z + \frac{1}{2\pi i} \frac{\bar{\phi}(z)}{\phi_0(z)} \]

\( q = \exp(2\pi i \psi) = c_2 z \exp \left( \frac{\bar{\phi}(z)}{\phi_0(z)} \right) \)

Constant because don't know \( \beta_1 \) exactly, only up to adding a multiple of \( \beta_0 \) can write power series....
Yukawa coupling in $H^{21}(\tilde{\gamma})$:

Let $W_k = \int_{x_2} \tilde{\gamma}(z) \wedge \frac{d^k}{dz^k} \tilde{\gamma}(z)$ (still same family of $\tilde{\gamma}$!).

Rewrite Picard-Fuchs in form $\frac{d}{dz^4} [\tilde{\gamma}] + \sum_{k=0}^3 C_k(z) \frac{d^k}{dz^k} [\tilde{\gamma}] = 0$

Then $W_4 + \sum_{k=0}^3 C_k W_k = 0$

But Griffiths tranversality $\Rightarrow \tilde{\gamma}$ is 1st, 2nd derivative have no $(0,3)$

$\Rightarrow W_0 = W_1 = W_2 = 0$.

Moreover:

$$0 = \frac{d^2 W_2}{dz^2} = \int \frac{d^2 \tilde{\gamma}}{dz^2} \wedge \frac{d^2 \tilde{\gamma}}{dz^2} + 2 \int \frac{d \tilde{\gamma}}{dz} \wedge \frac{d^3 \tilde{\gamma}}{dz^3} + \int \tilde{\gamma} \wedge \frac{d^4 \tilde{\gamma}}{dz^4}$$

$$= 0 + 2 \left( \frac{d W_3}{dz} - W_4 \right) + W_4$$

So $W_4 = 2 W_3'$, and get $W_3' + \frac{1}{2} C_3 W_3 = 0$!

Look at coefficient of $\frac{d^3}{dz^3}$ in Picard-Fuchs $\Rightarrow C_3(z) = \frac{3}{\bar{z}} - \frac{2 \cdot 5^5}{1 - 5^5 \bar{z}}$

$\Rightarrow (\text{log $W_3$})' = -\frac{3}{\bar{z}} + \frac{5^5}{1 - 5^5 \bar{z}}$. Integrating we get

$W_3(z) = \frac{C_1}{(2\pi i)^3 \frac{3}{\bar{z}} (5^5 \bar{z} - 1)}$. This is almost $\left< \frac{3}{\partial \bar{z}}, \frac{3}{\partial z}, \frac{3}{\partial \bar{z}} \right>$. Still need to normalize (want $\left< \cdots \right>$ rel. to $\tilde{\gamma}$, not $\tilde{\beta}$)

and switch to canonical coordinates (want $\left< \frac{3}{\partial w}, \frac{3}{\partial w}, \frac{3}{\partial \bar{w}} \right>$ not $\frac{3}{\partial z}$).

Normalization: scaling $\tilde{\gamma}$ by $f(z)$ changes $\left< \frac{3}{\partial \bar{z}}, \frac{3}{\partial z}, \frac{3}{\partial \bar{z}} \right> \sim f(z)^2$

(no derivative if $f$ come up because $\tilde{\gamma} \wedge \frac{3}{\partial z} \tilde{\gamma} \equiv 0$ for $c<3$)

In our case, want to scale by $\left< \frac{1}{\tilde{\beta}_0} \tilde{\gamma} \right> = \text{const.} \frac{1}{\phi_0(z)}$
\[ \langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \frac{c_1}{(2\pi)^3 \pi^3 (5^5 z - 1) \phi_0(z)^2} \]

Switching coordinates: \[ \frac{\partial}{\partial w} = (\frac{dw}{dz})^{-1} \frac{\partial}{\partial z} \Rightarrow \langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = \frac{c_1}{(5^5 z - 1) \phi_0(z)^2 \delta(z)^3} \]

where \( \delta(z) = 2\pi i \frac{dw}{dz} = z \frac{d\log q}{dz} = 1 + z \frac{d}{dz} \left( \frac{\phi(z)}{\phi_0(z)} \right) \)

Finally, we want to expand this as a power series in \( q \).

Since \( dq/dz = q \frac{d\log q}{dz} = \frac{q}{z} \delta(z) = c_2 \delta(z) \exp(\bar{\phi}/\phi_0) \), we have

\[ \frac{d^n}{dq^n} \langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = \left( \frac{1}{c_2 \delta(z) \exp(\bar{\phi}/\phi_0)} \frac{d}{dz} \right)^n \left( \frac{c_1}{(5^5 z - 1) \phi_0(z)^2 \delta(z)^3} \right) \]

To calculate expansion in \( q \) by evaluating here at \( z = 0 \) (from expansion of \( \phi(z), \bar{\phi}(z) \))

Get: \[ \langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = -c_1 - 575 \frac{c_1}{c_2} q - \frac{1950750}{6} \frac{c_2}{c_2^2} q^2 - \frac{10277690000}{c_2^3} q^3 \]

\[ - \frac{7458604825000}{24} \frac{c_1}{c_2^4} q^4 + \ldots \]

These coefficients are Gromov-Witten invariants of quintic.