① **Recall:** • $X_\psi := \left\{ (x_0 \ldots x_4) \in \mathbb{P}^4 \,/\, f_\psi = \sum_0^4 x_i^5 - 5\psi\, x_0 x_1 x_2 x_3 x_4 = 0 \right\}$

In affine chart $x_4 = 1$ (and where x_0, x_1, x_2 local coords., $x_3 = x_3(x_0, x_1, x_2)$):

$$\Omega_\psi = 5\psi \frac{dx_0 \wedge dx_1 \wedge dx_2}{\partial f_\psi / \partial x_3}\Big|_{X_\psi} = 5\psi \frac{dx_0 \wedge dx_1 \wedge dx_2}{(5x_3^4 - 5\psi x_0 x_1 x_2)}\Big|_{X_\psi}$$

• $G = \left\{ (a_0 \ldots a_4) \in \mathbb{Z}_5^5 \mid \sum a_i = 0 \right\} / \{(a,a,a,a,a)\} \simeq (\mathbb{Z}_5)^3$

acts by diagonal mult. by ξ^{a_i}, $\xi = e^{2\pi i/5}$

$\check{X}_\psi$ = crepant resolution of X_ψ / G

consider $\psi \to \infty$, i.e. $z = (5\psi)^{-5} \to 0$ (we'll see: LCSL)

Ω_ψ is G-invariant and induces $\check{\Omega}_\psi$ on $\check{X}_\psi$.

• **Residues:** Let $\bar{\Omega} = \sum_{i=0}^4 (-1)^i x_i \, dx_0 \wedge \ldots \widehat{dx_i} \ldots dx_4$ [in chart $x_4 = 1$: $\bar{\Omega} = dx_0 \wedge \ldots \wedge dx_3$]

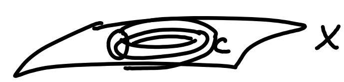
This is not a well-def^d 4-form on $\mathbb{P}^4$ because it's homogeneous of deg. 5 not 0

but if f, g homogeneous, $\deg f = \deg g + 5$, then $\frac{g\bar{\Omega}}{f}$ is a global meromorphic 4-form on $\mathbb{P}^4$, with poles where $f = 0$.

Ex: $\frac{5\psi \bar{\Omega}}{f_\psi}$ = 4-form with poles along X_ψ.

Now, if we have a 4-form with poles along X, it has a residue on X – ideally a 3-form on X, or at least a class in $H^3(X, \mathbb{C})$.

$\mathrm{Res}_X\left(\frac{g\bar{\Omega}}{f}\right)$ s.t. $\forall$ 3-cycle C in X,



let Γ = "tube" 4-cycle = preimage of C in ∂(tub. nbd)

then $\frac{1}{2\pi i} \int_\Gamma \frac{g\bar{\Omega}}{f} = \int_C \mathrm{Res}_X\left(\frac{g\bar{\Omega}}{f}\right)$.

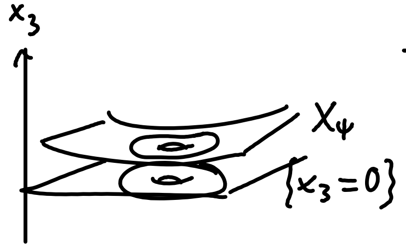
If we have simple poles along $X = f^{-1}(0)$, can get a 3-form: charact^d by

$$\mathrm{Res}_X\left(\frac{g\bar{\Omega}}{f}\right) \wedge df = g\bar{\Omega} \text{ at every point of } X$$

Here: $\Omega_\psi = \mathrm{Res}_{X_\psi}\left(\frac{5\psi\bar{\Omega}}{f_\psi}\right)$ (compare w/ formula above)

(in local coords, X_ψ: $x_3 = x_3(x_0, x_1, x_2)$: $\mathrm{Res}_{x_3}\left(\frac{5\psi\, dx_0 dx_1 dx_2 dx_3}{f_\psi}\right) = \frac{5\psi\, dx_0\, dx_1\, dx_2}{(\partial f_\psi / \partial x_3)}$).

② • We're interested in torus $\check{T}_0 = T_0/G \subset \check{X}_\psi$ (survives degeneration $\Rightarrow \beta_0 = [\check{T}_0]$ monodromy-invt)



$$T_0 : \begin{cases} x_4 = 1, \quad |x_0| = |x_1| = |x_2| = \delta, \\ x_3 = \text{the root of } f_\psi = 0 \text{ which tends to } 0 \text{ as } \psi \to \infty \end{cases}$$

Let $T^4 = \{ |x_0| = |x_1| = |x_2| = |x_3| = \delta, \ x_4 = 1 \} \subset \mathbb{P}^4$ ("tube" 4-cycle for T_0)

Then residue formula $\Rightarrow \int_{T_0} \Omega_\psi = \frac{1}{2\pi i} \int_{T^4} \frac{5\psi \, dx_0 \, dx_1 \, dx_2 \, dx_3}{f_\psi}$

$$= \frac{1}{2\pi i} \int_{T^4} \frac{dx_0 \, dx_1 \, dx_2 \, dx_3}{(5\psi)^{-1}(x_0^5 + x_1^5 + x_2^5 + x_3^5 + 1) - x_0 x_1 x_2 x_3}$$

$$= \frac{-1}{2\pi i} \int_{T^4} \frac{dx_0 \, dx_1 \, dx_2 \, dx_3}{x_0 x_1 x_2 x_3} \frac{1}{1 - \frac{(x_0^5 + \dots + x_3^5 + 1)}{5\psi \, x_0 x_1 x_2 x_3}}$$

$$= \frac{-1}{2\pi i} \sum_{m=0}^{\infty} \int_{T^4} \frac{dx_0 \, dx_1 \, dx_2 \, dx_3}{x_0 x_1 x_2 x_3} \underbrace{\frac{(x_0^5 + \dots + x_3^5 + 1)^m}{(5\psi)^m (x_0 x_1 x_2 x_3)^m}}_{(\#)}$$

Now use again residues $\rightsquigarrow$ only terms contributing are terms in (#) not involving any x_i's $\rightsquigarrow$ need term $(x_0^{5n} x_1^{5n} x_2^{5n} x_3^{5n})$ in numerator expansion for $m = 5n$. ie. need to pick each of $x_0^5, \dots, x_3^5, 1$ n times.
such terms is $(5n)!/(n!)^5$ (e.g. choose x_0's $\binom{5n}{n}$ then x_1's $\binom{4n}{n} \dots x_3$'s $\binom{2n}{n}$)

Hence $\int_{T_0} \Omega_\psi = -(2\pi i)^3 \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}$

or in terms of $z = (5\psi)^{-5}$, proportional to $\phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$

• This tells us how to scale $\check{\Omega}_\psi$ for normalization.
Then we need <u>another</u> period (along what 3-cycle?) to get canonical coordinate.

③

Instead of brute-force calc$^{\underline{n}}$, use a general fact: $\underline{\underline{\text{all}}}$ periods $\int_C \check{\Omega}_\psi$ satisfy a same differential equation - the Picard-Fuchs eq$^{\underline{n}}$ for $\{\check{X}_\psi\}$

* Guess for Picard-Fuchs eqn: $\phi_0(z) = \sum a_n z^n$, $a_n = \dfrac{(5n)!}{(n!)^5}$

obey recurrence relation $(n+1)^5 a_{n+1} = \dfrac{(5n+5)!}{(n!)^5} = (5n+5)(\ldots)(5n+1)\, a_n$

or $(n+1)^4 a_{n+1} = 5(5n+1)\ldots(5n+4)\, a_n$

Let $\Theta = z\dfrac{d}{dz}$: $\Theta\left(\sum c_n z^n\right) = \sum n c_n z^n$ so ϕ_0 solves

$$\Theta^4 \phi = 5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\phi$$

Prop: $[\check{\Omega}_\psi]$ and hence all periods $\int_C \check{\Omega}_\psi$ also satisfy this equation

Rmk: Simple reason why all periods satisfy some diff-equation:

$H^3(\check{X}_\psi)$ is 4-dimensional, so $[\check{\Omega}_\psi], \left[\dfrac{\partial\check{\Omega}}{\partial\psi}\right], \ldots, \left[\dfrac{\partial^4\check{\Omega}}{\partial\psi^4}\right]$ must be linearly related

$\Rightarrow$ so are their $\int$ over any 3-cycle.

$\Rightarrow \int_C \check{\Omega}_\psi$ solves 4^{th} order diff-eq$^{\underline{n}}$!

Proof sketch: Recall: $\Omega_\psi = \mathrm{Res}_{X_\psi}\left(\dfrac{5\psi\,\overline{\Omega}}{f_\psi}\right)$

$\rightsquigarrow$ differentiating k times, $\dfrac{\partial^k}{\partial\psi^k}\Omega_\psi = \mathrm{Res}_{X_\psi}\left(\dfrac{g_k\,\overline{\Omega}}{f_\psi^{k+1}}\right)$

so... compute $\Theta^4\Omega_\psi$ and $5z(5\Theta+1)\ldots(5\Theta+4)\Omega_\psi$ where

$\Theta = z\dfrac{d}{dz} = \dfrac{1}{5}\psi\dfrac{d}{d\psi}$ in each form. Then show residues equal.

To compare residues of forms with order 5 poles along X_ψ, need algorithm for pole order reduction [Griffiths]:

Namely φ 3-form w/ poles of order l along $X_\psi \Rightarrow$ can write

$$\varphi = \frac{1}{f_\psi^l}\sum_{i<j}(-1)^{i+j}(x_i g_j - x_j g_i)\, dx_0 \wedge \ldots \widehat{dx_i}\,\widehat{dx_j}\ldots dx_4 \quad (g_0 \ldots g_4 \text{ degree } 5l-4)$$

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$$\Rightarrow d\varphi = \frac{1}{f_\psi^{l+1}}\left(l\sum_j g_j \frac{\partial f_\psi}{\partial x_j} - f_\psi \sum_j \frac{\partial g_j}{\partial x_j}\right)\Omega$$

so $\left(\sum g_j \frac{\partial f_\psi}{\partial x_j}\right)\frac{\Omega}{f_\psi^{l+1}}$ can be rewritten as (lower order pole) + $\underset{\text{doesn't affect residue.}}{(\text{exact})}$

criterion: top order term $\in$ Jacobian ideal gen^d by $\frac{\partial f_\psi}{\partial x_j}$'s. $\Rightarrow$ can reduce.

Apply pole order reduction to $\Theta^4\Omega_\psi - 5z(5\Theta+1)\dots(5\Theta+4)\Omega_\psi$, show $[\mathrm{Res}]=0$. (easier with computer algebra software). $\blacksquare$

Now we can find other periods of $\check{\Omega}_\psi$ using the theory of <u>diff^l equation with regular singular pts</u> = diff. eqn. of the form

$$\Theta^s f + \sum_{j=0}^{s-1} B_j(z)\Theta^j f = 0 \quad \text{where } \Theta = z\frac{d}{dz},\ B_j \text{ holomorphic at } z=0.$$

- Reduce to a 1^st order system: let

$$A(z) = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & & \ddots & \\ & & 0 & 1 \\ -B_0(z) & \cdots & & -B_{s-1}(z) \end{pmatrix}, \quad w(z) = \begin{pmatrix} f(z) \\ \Theta f(z) \\ \vdots \\ \Theta^{s-1} f(z)\end{pmatrix}$$

Then eqn becomes $\Theta w(z) = A(z)w(z)$.

<u>Thm:</u> $\exists$ constant $s\times s$ matrix R and $s\times s$ matrix of holom. functions $S(z)$ s.t.

$$\Phi(z) = S(z)\exp((\log z)R) = S(z)\left(Id + (\log z)R + \frac{(\log z)^2}{2}R^2 + \dots\right)$$

is a fundamental system of sol^ns for $\Theta w(z) = A(z)w(z)$.

Moreover, if $A(0)$ doesn't have eigenvalues differing by a nonzero integer then can take $R = A(0)$.

<u>NB:</u> Φ is <u>multivalued</u>! $z\mapsto e^{2\pi i}z$ gives $\Phi(z)\mapsto\Phi(z)e^{2\pi i R}$

so the monodromy is $e^{2\pi i R}$

(5) In our case: $\mathcal{D}\phi = \Theta^4\phi - 5z(5\Theta+1)\dots(5\Theta+4)\phi = 0$

↑ coeff of Θ^4 is $1-5^5 z$

Coeff's of $\Theta^{i\le 3}$ are cst. z

eqn rewrites as: $\Theta^4\phi - \dfrac{5z}{1-5^5 z} P_3(\Theta)\phi = 0$

↑ indept of z

This is of the desired form, and $A(0) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ for $z=0$.

nilpotent $\Rightarrow$ assumption satisfied.

So the monodromy is given by $T = e^{2\pi i A(0)}$ unipotent of max. order

$$= \begin{pmatrix} 1 & 2\pi i & \frac{(2\pi i)^2}{2} & (2\pi i)^3/6 \\ & \ddots & \ddots & (2\pi i)^2/2 \\ & & \ddots & 2\pi i \\ & & & 1 \end{pmatrix}$$

In particular: 1^{st} column of Φ is inv't under $\Phi \mapsto \Phi T$: single valued sol.

the others are multivalued. (2^{nd} column $\mapsto$ itself $+ (2\pi i)\cdot(1^{st}$ col.))

<u>Relevance</u>: if $\omega(z) = \int_\beta \Omega$ is a period then it's a sol^{n} to Picard-Fuchs

$\Rightarrow$ it's a linear combination of fund^{l} solutions

= first row of matrix $\Phi(z)$

so $\exists$ basis $\alpha_1 \dots \alpha_4$ of $H_3(\check{X}, \mathbb{C})$ s.t. $\int_{\alpha_i}\Omega = \Phi(z)_{1i}$

The monodromy transformation in this basis is then

$T = \exp(2\pi i A(0))$ — this proves $z=0$ is max. unipotent (LCSL)