① • Recall from last lecture: Large cx. structure limit degenerations

⤳ <u>Canonical coordinates</u> on complex moduli space

we have a basis $(\alpha_0 \dots \alpha_s, \beta_0 \dots \beta_s)$ of $H_3(X,\mathbb{Z})$

β_0 invariant under monodromy, $\beta_1 \dots \beta_s$ mapped by $\beta_i \overset{\gamma_j}{\longmapsto} \beta_i + \underset{\in\mathbb{Z}}{m_{ji}} \beta_0$

Normalize $\int_{\beta_0} \Omega = 1$: then $q_i := \exp(2\pi i \int_{\beta_i} \Omega)$ canonical coords.

<u>Ex:</u> for family of tori with monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\int_a \Omega = 1$, $\int_b \Omega = \tau$

$$q = \exp(2\pi i \tau)$$

• On mirror: $e_1^\vee \dots e_s^\vee$ basis of $H_2(\check{X}, \mathbb{Z})$

⤳ coords. on complexified Kähler moduli space : $\check{q}_i = \exp(2\pi i \int_{e_i^\vee} B + i\omega)$

<u>Ex:</u> for T^2, $\check{q} = \exp(2\pi i \cdot \int_{T^2} B + i\omega)$

<u>Mirror symmetry statement:</u>

$f: \mathcal{X} \to (D^*)^s$ family of CY 3-folds with LCSL point at 0.

Then $\exists$ CY 3-fold $\check{X}$ + $\exists$ choice of bases $\alpha_0 \dots \alpha_s, \beta_0 \dots \beta_s$ on $H_3(X)$

$e_1 \dots e_s$ on $H^2(\check{X})$

s.t. under the map $m: \mathcal{M}_{cx}(X) \longrightarrow \mathcal{M}_{käh}(\check{X})$ in canonical coordinates

$$(q_1 \cdots q_s) \longmapsto (\check{q}_1 \cdots \check{q}_s) = (q_1 \dots q_s)$$

the Yukawa Couplings $\left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right\rangle_p = \left\langle \frac{\partial}{\partial \check{q}_i}, \frac{\partial}{\partial \check{q}_j}, \frac{\partial}{\partial \check{q}_k} \right\rangle_{m(p)}$

(2,1) Yukawa coupling: $\int_X \Omega \wedge \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} \Omega$

at point $p \in \mathcal{M}_{cx}$ given by periods q_i

(recall Kodaira-Spencer map

$$\frac{\partial}{\partial q_i} \in T\mathcal{M}_{cx} \xrightarrow{\simeq} H^{2,1} \ni \left[\frac{\partial \Omega}{\partial q_i}\right]^{(2,1)})$$

(1,1) Yukawa Coupling

ie. 3-pt GW invts.

involving $\frac{\partial}{\partial \check{q}_i} \in T\mathcal{M}_{käh}$

$\downarrow \simeq$

$\frac{\partial}{\partial \check{q}_i}([B + i\omega]) \in H^{1,1}$

(Equivalent statement: 2 algebra structures $(H^*(X, \Lambda^* TX), \wedge)$ and $(QH^*(\check{X}), *)$ match).

$(\cong H^{n-*,*}(X))$

② <u>Now:</u> Computing canonical coords & (2,1)-coupling on mirror quintics.

<u>Recall:</u> $X_\psi := \{ (x_0 \dots x_4) \in \mathbb{P}^4 \,/\, f_\psi = \sum_0^4 x_i^5 - 5\psi\, x_0 x_1 x_2 x_3 x_4 = 0 \}$

$G = \{ (a_0 \dots a_4) \in \mathbb{Z}/5^5 \mid \sum a_i = 0 \} / \{(a,a,a,a,a)\} \simeq (\mathbb{Z}/5)^3$

acts by diagonal mult. by ξ^{a_i}, $\xi = e^{2\pi i/5}$.

$\check{X}_\psi$ = crepant resolution of X_ψ/G (singo. $\overline{C_{ij}} = \{x_i = x_j = 0\} \simeq \mathbb{P}^1$ intersecting at pts P_{ijk}).

has $h^{1,1} = 101$, $h^{2,1} = 1$.

- Note: $(x_0 \dots x_4) \mapsto (\xi^a x_0, x_1 \dots x_4)$ induces $X_\psi \cong X_{\xi\psi}$ hence $\check{X}_\psi \cong \check{X}_{\xi\psi}$

$\Rightarrow$ to get an actual coord. on moduli space, let $z = (5\psi)^{-5}$

★ $z \to 0$ ie. $\psi \to \infty$ corresponds to <u>toric degeneration</u> of X_ψ to $x_0 x_1 x_2 x_3 x_4 = 0$ union of 5 coordinate hyperplanes

This is maximally unipotent, and hence a LCSL degeneration!

Need to compute the periods of Ω a holom. volume form on $\check{X}_\psi$.
Because Ω on $\check{X}_\psi$ is induced by G-inv. holom. vol. form on X_ψ,
(quotient by G, then pullback via resolution map $\pi: \check{X}_\psi \to X_\psi/G$)
we can work on X_ψ instead.

$\exists$ <u>Explicit method</u> (Candelas - de la Ossa - Greene - Parkes)
[could also use lots of toric geometry ... see Cox - Katz book].

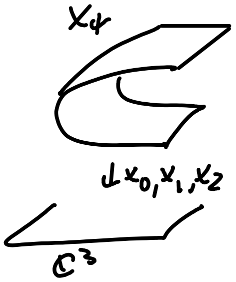
- <u>holom volume form:</u>

in affine subset $x_4 = 1$, let Ω_ψ be 3-form on X_ψ characterized by
$$\Omega_\psi \wedge df_\psi = 5\psi\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \quad \text{at every point of } X_\psi$$

E.g. if we're at a point where $\frac{\partial f_\psi}{\partial x_3} \neq 0$, then can use x_0, x_1, x_2 as local coords on X_ψ & write Ω_ψ as a scalar mult. of $dx_0 \wedge dx_1 \wedge dx_2$.

③

Necess: $\Omega_\psi = 5\psi \dfrac{dx_0 \wedge dx_1 \wedge dx_2}{\partial f_\psi / \partial x_3}\Big|_{X_\psi} = 5\psi \dfrac{dx_0 \wedge dx_1 \wedge dx_2}{(5x_3^4 - 5\psi x_0 x_1 x_2)}\Big|_{X_\psi}$



This doesn't have poles where $\dfrac{\partial f_\psi}{\partial x_3} = 0$ because there the Jacobian of proj^n to $(x_0 x_1 x_2)$ coords. vanishes, ie. numerator also has a zero.

formulas e.g. in terms of $x_0 x_1 x_3$ etc. still make sense ✓

- There isn't a zero or pole either at $x_4 = 0$; e.g switch to chart $x_3 = 1$ by setting $\tilde{x}_i = \dfrac{x_i}{x_3}$, then check that Ω_ψ still looks same in new coords.

or: divisor def^d by Ω_ψ is a multiple of $\{x_4 = 0\}$, but linearly ~ 0 since X_ψ Calabi-Yau

* Ω_ψ is G-invariant & induces a holom. volume form on $(X_\psi/G)^{\text{nonsing}}$

pull back via resolution $\rightsquigarrow$ holom. vol. form $\check{\Omega}_\psi$ on $\check{X}_\psi$

(extends across exc. divisors of blowups because resol^n is crepant)

- 3-cycle β_0 in $\check{X}_\psi$:

for $z=0$, $\{\prod x_i = 0\} \supset$ tori T^3 in components $\mathbb{P}^3$:

$$T_0 = \{(x_0 \dots x_4) \in \mathbb{P}^4 \,/\, x_4 = 1,\ |x_0| = |x_1| = |x_2| = \delta,\ x_3 = 0\}$$

for $z \neq 0$: take $x_4 = 1$, $|x_0| = |x_1| = |x_2| = \delta$,

$x_3 =$ the root of $f_\psi = 0$ which tends to 0 as $\psi \to \infty$

namely let $x_3 = (\psi x_0 x_1 x_2)^{1/4}\, y$ $\rightsquigarrow$

$$f_\psi = 0 \iff x_0^5 + x_1^5 + x_2^5 + (\psi x_0 x_1 x_2)^{5/4} y^5 + 1 - 5(\psi x_0 x_1 x_2)^{5/4} y = 0$$

ie. $y = \dfrac{y^5}{5} + \dfrac{1 + x_0^5 + x_1^5 + x_2^5}{(\psi x_0 x_1 x_2)^{5/4}}$

as $\psi \to \infty$, one root $y \sim \psi^{-5/4}$, the 4 others $\to \sqrt[4]{5}$

$x_3 \sim \psi^{-1}$, the 4 others $\sim \psi^{1/4}$

hence $\exists$ well-defined branch of x_3 that $\to 0$.

This defines a 3-torus T_0 in X_ψ,

(4) G acts on T_0, freely ($T_0 \cap C_{ij} = \phi$) $\rightsquigarrow$ torus $\check{T}_0$ in X_ψ/G and $\check{X}_\psi$

* Because $\check{T}_0$ still makes sense as smooth submfld for $z=0$, its class $\beta_0 = [\check{T}_0] \in H_3(\check{X}_\psi, \mathbb{Z})$ is preserved by the monodromy.

($\beta_0 \in W_0$ for weight filtration!)

* we want to compute $\int_{\check{T}_0} \check{\Omega}_\psi$ or equivtly up to 5^3, $\int_{T_0} \Omega_\psi$

<u>Tool:</u> residue formula $\frac{1}{2\pi i} \int_{S^1} f(z)\,dz = \sum_{\substack{z_i \text{ pole of } f \\ z_i \in D^2}} \operatorname{Res}_{z_i}(f)$

So: let $T^4 = \{ |x_0| = |x_1| = |x_2| = |x_3| = \delta,\ x_4 = 1 \} \subset \mathbb{P}^4$:

$$\frac{1}{2\pi i} \int_{T^4} 5\psi \frac{dx_0\, dx_1 dx_2\, dx_3}{f_\psi} = \int_{T^3_{x_0x_1x_2}} \left(\frac{1}{2\pi i} \int_{S^1_{x_3}} 5\psi \, \frac{dx_3}{f_\psi} \right) dx_0 dx_1 dx_2$$

we've seen only one root of f_ψ near 0

$\Rightarrow$ only one pole, on T_0! residue $= \dfrac{5\psi}{\partial f_\psi / \partial x_3}$

Thus $\ldots = \displaystyle\int_{T_0} \frac{5\psi\, dx_0\, dx_1\, dx_2}{\partial f_\psi / \partial x_3} = \int_{T_0} \Omega_\psi.$

So $$\int_{T_0} \Omega_\psi = \frac{1}{2\pi i} \int_{T^4} \frac{dx_0\, dx_1\, dx_2\, dx_3}{(5\psi)^{-1}(x_0^5 + x_1^5 + x_2^5 + x_3^5 + 1) - x_0 x_1 x_2 x_3}$$