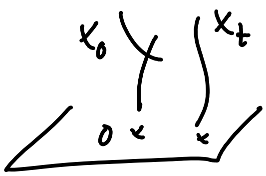
① Degeneration & monodromy: (linear algebra)



$$\begin{tikzcd} \mathcal{X} \arrow[d] & \supset & X_t \arrow[d] \\ D^2 & \ni & t \end{tikzcd}$$

family of compact Kähler manifolds,
X_t smooth, X_0 singular
(or just consider family over $D^2 - 0$!)

We've seen: monodromy around $t=0$ induces $\varphi_* \in \mathrm{Aut}(H^n(X_{t_0}, \mathbb{Z}))$

- replacing φ by φ^N ("base change": $X'_t = X_{t^N}$), can assume φ_* is unipotent ie. $(\varphi_* - id)^k = 0$; maximally unipotent := $k = n+1$.

- Can define a weight filtration associated to unipotent φ_*:
[Comes from Jordan block decomposition of φ_*]

let $N = \log(\varphi_*) = (\varphi_* - id) - \frac{(\varphi_* - id)^2}{2} + \dots + (-1)^{n+1} \frac{(\varphi_* - id)^n}{n}$

nilpotent $N^{n+1} = 0$ acting on $V = H^n(X, \mathbb{Q})$

$\Rightarrow \exists!$ filtration $0 \subseteq W_0 \subseteq W_1 \subseteq \dots \subseteq W_{2n} = V$ s.t.

$$\begin{cases} N(W_i) \subseteq W_{i-2} \\ N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1} \quad \forall k \end{cases}$$
(basic linear algebra)

Ex: for the elliptic curve last time, $\varphi = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array} = \exp \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{N}$ $(n=1)$

$(\varphi - Id)^2 = 0$

$$0 \subseteq W_0 \subseteq W_1 \subseteq W_2 = H^1(C, \mathbb{Q}) \quad (\simeq \mathbb{Q}^2)$$

$\overset{\uparrow}{\mathrm{Im}\, N =} \ \overset{\uparrow}{\ker N} = \mathrm{span}(a)$ = direction invariant by monodromy.

- Note: if $N =$ Jordan block $\underbrace{\overset{e_1 \dots e_{k+1}}{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}}}_{(k+1) \times (k+1)}$ $(k \leq n)$ then $W_{n-k} = W_{n-k+1} = \mathrm{span}(e_1)$

$W_{n-k+2} = W_{n-k+3} = \mathrm{span}(e_1, e_2)$

. . . . .

$W_{n+k} = W_{n+k+1} = \dots = \mathrm{span}(e_1 .. e_{k+1})$

$\rightsquigarrow$ explicitly relate weight filtration $\leftrightarrow$ Jordan decomposition.

* In fact, the interplay of weight filtration with Hodge filtration
$F^p = H^{n,0} \oplus \dots \oplus H^{p,n-p}$ $\quad (H^n = F^0 \supseteq F^1 \supseteq \dots, \quad F^p/F^{p+1} \cong H^{p,n-p})$
gives notion of "mixed Hodge structure". We won't say more about those.
(Point: $\exists$ limiting Hodge filtration at $t \to 0$ [Schmid])

* Now consider a multidimensional family

$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ (D^2)^s \end{tikzcd}$$

smooth over $(D^*)^s$ $\quad (D^* = D^2 - \{0\})$

then we have s monodromies $\varphi_1, \dots, \varphi_s \in \mathrm{Aut}\, H_n(X)$,
$[\varphi_i, \varphi_j] = 0$ $\quad$ (since $\pi_1((D^*)^s) = \mathbb{Z}^s$ abelian)
$\rightsquigarrow$ $N_i = \log \varphi_i$ also commute.

Thm (Cattani-Kaplan)

> All elements of the form $\sum \lambda_i N_i$, $\lambda_i > 0$ have the same monodromy weight filtration.

Want to consider a "universal family" of CY near a "deepest corner" :=
"Large complex structure limit point" in moduli space

Def: (Morrison)

$\mathcal{X} \to (D^*)^s \subset (D^2)^s$ family of CY n-folds, $s = h^{n-1,1}(X)$
s.t. Kodaira-Spencer map $T_t((D^*)^s) \to H^1(TX_t)$
is an isomorphism at every point of $(D^*)^s$

We say $0 \in (D^2)^s$ is a *large complex str. limit point* (LCSL point)
if
(1) the monodromies φ_j around each factor are all unipotent
(2) let $N_j = \log \varphi_j$, and $N = \sum \lambda_j N_j$ for $\lambda_j > 0$ arbitrary.
Then weight filtration $0 \subseteq W_0 \subseteq W_1 \subseteq W_2 \subseteq \dots \subseteq W_{2n} = H^n(X, \mathbb{Q})$
has $\dim W_0 = \dim W_1 = 1$, $\dim W_2 = \dim W_3 = s+1$.
(3) let α_0^* generator of W_0, then $\exists$ $\mathbb{Q}$-basis $(\alpha_0^*, \alpha_1^*, \dots \alpha_s^*)$ of W_2
st. $N_j(\alpha_k^*) = \delta_{jk} \alpha_0^*$

This essentially says: → family is locally "full" deformation $(s = h^{2,1})$

→ $W_0 = W_1$ rank 1 := singles out 1-dim'l subspace of $H^n(X)$ $\mathrm{span}(\alpha_0^\alpha)$ preserved by the whole monodromy.

→ W_2 dim. s & invertibility means: for each factor D^2 we get a class α_j^α s.t. $\varphi_j(\alpha_j^\alpha) = \alpha_j^\alpha + \alpha_0^\alpha$ & α_j^α inv't under other φ_i

<u>Fact:</u> ‖ if $h^{n-1,1} = s = 1$ then this is equivalent to: monodromy around 0 is maximally unipotent

<u>Ex:</u> family of elliptic curves seen last time is a LCSL point.

★ Now, for a family of CY 3-folds:

by def^n,

$$0 \subset \underbrace{W_0 = W_1}_{\text{dim. } 1} \subset \underbrace{\underset{\uparrow}{W_2} = W_3}_{\text{dim } s+1 = h^{2,1}+1} \subset \underbrace{W_4 = W_5}_{\text{dim. } 2s+1} \subset \underbrace{W_6 = H^3(X,\mathbb{Q})}_{\text{dim. } 2s+2}$$

$\uparrow$

use $N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1}$ to get dim. W_3, W_4, W_5

★ $H^3(X)$ carries intersection pairing $(\cdot,\cdot)$, preserved by φ_*

⇒ $N = \log \varphi_*$ is in the Lie algebra: $(x, Ny) + (Nx, y) = 0$.

<u>Lemma:</u> ‖ $W_{4-2i} = W_{2i}^{\perp}$

SKIP

<u>Pf:</u> • $\left.\begin{array}{l} W_0 = \mathrm{Im}\, N^3 \\ W_4 = W_5 = \ker N^3 \end{array}\right\}$ ⇒ if $x \in W_4$, $N^3 y \in W_0$ then $(x, N^3 y) = -(N^3 x, y) = 0$.

+ dimensions match

• $N(W_4) = W_2$ (onto since: $N : W_4/W_{3=W_2} \xrightarrow{\sim} W_2/W_{1=W_0}$

and: $W_0 = \mathrm{Im}\, N^3 = N(\underbrace{\mathrm{Im}\, N^2}_{\subset W_4})$)

⇒ if $x, Ny \in W_2$ $(y \in W_4)$ then

$(x, Ny) = -(Nx, y) = 0$ (since $W_0 \perp W_4$)

+ dims match. ▲

- Passing to $H_3(X;\mathbb{Q})$ by Poincaré duality, let $S_i =$ P.D. of W_i
  or equivalently, viewing $H_3 \simeq (H^3)^*$, $S_i =$ annihilator of W_{4-2i}

Prop: Given LCSL point in moduli space of CY 3-folds w/ $h^{2,1}=s$,
$\exists$ $\mathbb{Z}$-basis $\alpha_0,\dots\alpha_s,\ \beta_0,\dots,\beta_s$ of $H_3(X)$ s.t.
$\beta_0 \in S_0$, $\beta_1..\beta_s \in S_2$, $\alpha_1..\alpha_s \in S_4$, $\alpha_0 \in S_6 = H_3(X)$
s.t. $(\alpha_i,\alpha_j) = (\beta_i,\beta_j) = 0$, $(\alpha_i,\beta_j) = \delta_{ij}$.
\+ by def. of S_2: $\varphi_j(\beta_0) = \beta_0$, $\varphi_j(\beta_i) = \beta_i + m_{ji}\beta_0$ for some $m_{ji} \in \mathbb{Z}$

skip

Pf:
- $\beta_0 :=$ $\mathbb{Z}$-generator of S_0 (unique up to sign)
- extend to a $\mathbb{Z}$-basis of S_2;
  by lemma, S_2 is Lagrangian wrt $(\cdot,\cdot)$ so $(\beta_i,\beta_j)=0$
- let $\beta_i^* =$ dual basis of $S_2^* = H^3/W_2$ ie. $\beta_i^*(\beta_j) = \delta_{ij}$
  $\alpha_i \in H_3 =$ Poincaré dual of some lift of β_i^* to H^3 $\Rightarrow (\alpha_i,\beta_j) = \delta_{ij}$
  can ensure $(\alpha_i,\alpha_j)=0$ inductively by $\alpha_i \leftarrow \alpha_i - \sum(\alpha_i,\alpha_j)\beta_j$
- $\alpha_1\dots\alpha_s \in S_4$ since $(\alpha_i,\beta_0)=0 \Rightarrow \alpha_i \in S_0^{\perp}$

∎

<u>Canonical coordinates</u>: given $\mathfrak{X} \to (\mathbb{D}^*)^s$ LCSL,

let $\Omega(t_1..t_s) =$ holom. vol. form on $X_{(t_1..t_s)}$, normalized so that
$\int_{\beta_0} \Omega(t_1..t_s) = 1$. Then $w_i(t_1\dots t_s) := \int_{\beta_i} \Omega(t_1\dots t_s)$

Not quite a coordinate because of monodromy:
as t_j goes around the origin, $\beta_i \longmapsto \varphi_j(\beta_i) = \beta_i + m_{ji}\beta_0$
so $w_i \longmapsto w_i + m_{ji}$.

instead set $q_i = \exp(2\pi i\, w_i)$ well-defined functions on $(\mathbb{D}^*)^s$
<u>canonical coordinates</u> (canonical only once basis $\{\beta_i\}$ is chosen!)

(q_i has zero of order m_{ji} along $t_j=0$)

if m_{ji}'s are nonnegative then get coords. on $(D^2)^s$ ...
choose basis of S_2 appropriately!

Ex: for elliptic curve of last week, $q = \exp(2\pi i \tau(t))$, $\tau(t) = \int_b \Omega$
where $\int_a \Omega = 1$.

Last time, saw e_i basis of $H^2(\check{X}, \mathbb{Z})$, $e_i \in$ Kähler cone

⇝ coords. on complexified Kähler moduli space:

if $[B+i\omega] = \sum \check{t}_i e_i$, let $\check{q}_i = \exp(2\pi i \check{t}_i) \in \mathbb{C}^*$

(ie. $\check{t}_i = \int_{e_i^*} B+i\omega$)

Ex: for T^2, $\check{q}_i = \exp(2\pi i \cdot \int_{T^2} B+i\omega)$

Mirror symmetry statement:

> $f: \mathcal{X} \to (D^*)^s$ family of CY 3-folds with LCSL point at 0.
> Then $\exists$ CY 3-fold $\check{X}$ + $\exists$ choice of bases $\alpha_0 \dots \alpha_s, \beta_0 \dots \beta_s$ on $H_3(X)$
> $e_1 \dots e_s$ on $H^2(\check{X})$
>
> s.t. under the map $m: (D^*)^s \longrightarrow \mathcal{M}_{K\ddot{a}h}(\check{X})$
> $(q_1 \cdots q_s) \longmapsto (\check{q}_1 \cdots \check{q}_s)$ in canonical coordinates
>
> the Yukawa couplings $\left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right\rangle_p = \left\langle \frac{\partial}{\partial \check{q}_1}, \frac{\partial}{\partial \check{q}_2}, \frac{\partial}{\partial \check{q}_3} \right\rangle_{m(p)}$

(2,1) Yukawa coupling: $\int_X \Omega \wedge \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} \Omega$
at point p given by periods q_i

(1,1) Yukawa coupling
ie. GW invts.

$2\pi i\, \check{q}_i \frac{\partial}{\partial \check{q}_i} = \frac{\partial}{\partial \check{t}_i} = e_i \in H^2(\check{X}, \mathbb{Z})$

Next time: computing canonical coords & (2,1)-coupling on mirror quintics.