* Schedule: TuTh 9:30-11 in 31 Evans
* Web page: math.berkeley.edu/~auroux/277F09/

**Goal:** overview of various aspects of mirror symmetry.

What is mirror symmetry? ... a phenomenon? 
  a conjecture? 
  a philosophy? ...

0. The physical origins: [le the only physics in this class!!]

  - *Supersymmetric string theories:
    propagation of a string in spacetime $\rightarrow$ surface $\Sigma$ ("worldsheet")
    $\rightarrow$ 2D quantum field theory (on $\Sigma$), where some fields take values in manifolds (space-time)
    Require this theory to be supersymmetric (various operators act on the space of states preserving all observables) and conformal (under conformal change of metric on $\Sigma$).

  - Most common instance of such a theory: "nonlinear $\sigma$-model"
determined by data: = a Calabi-Yau manifold
    
    - $(X, J)$ smooth complex manifold 
      (ideally: compact, dim $\geq 3$, maybe even $b_1 = 0$)
    - $K_X := \wedge^n T_X \cong \mathcal{O}_X$, so $\exists \Omega \in H^0(X, K_X) = \Omega^{n,0}(X)$
      nonvanishing holomorphic volume form
    - $\omega^C = B + i\omega \in \Omega^{1,1}(X)$ complexified Kähler form:
      (closed, $\text{Im} \omega^C = \omega$ nondegenerate).

    Deformations of the theory:
    \[
    \begin{align*}
    \text{deformation of } & J : H^2(X, TX) \\
    \text{deformation of } & \omega^C : H^{1,1}(X) = H^1(X, \Omega^1_X)
    \end{align*}
    \]

In fact, $\exists$ supersymmetry operators $Q, \bar{Q}$ ("charges") whose simult. eigenspaces are $H^q(\wedge^p TX)$ and $H^q(X, \Omega^p_X)$.
Moreover, upon changing labelling of supersymmetry operators \((q,\bar{q}) \leftrightarrow (-q, \bar{q})\) should get a new sigma-model in which eye-sympos are exchanged... can't do this on the same manifold, but hope \(\mathbb{E}(X^r, J^r, \omega^r)\) Calabi-Yau st.

\[ H^q(X, \Lambda^p T X) \cong H^q(X^r, J^r, \omega^r) \text{ & vice versa} \]

This is super-conformal field theories are equivalent

Then say \((X, J, \omega^c)\) & \((X^r, J^r, \omega^r)\) are mirror.

There's two "topologically twisted" variants, "A-model" and "B-model"

- A-model only depends on \(\omega^c\), not \(J\) — symplectic geometry
- B-model only depends on \(J\), not \(\omega^c\) — complex geometry

Then A-model on \((X, \omega^c)\) is equivalent to B-model on \((X^r, J^r)\)

B-model \((X, J)\) A-model \((X^r, \omega^r)\)

For this to become a mathematical statement, need some consequences of equivalence of SCFTs that can be defined in mathematical terms!

**Answer:** "correlation functions" (= expectation values for observables ...)

will have a mathematical formulation

(SFTs themselves involve integral over \(\text{Map}(\mathbb{E}, X)\) ??)

---

1. Hodge theory, quantum cohomology:

- First thing we expect of mirror CYs: \(H^q(X, \Lambda^p T X) \cong H^q(X^r, J^r, \omega^r)\)

\(\text{CY assumption} \Rightarrow \Lambda^p T X \xrightarrow{\sim} J^r \otimes \omega^r\)

\(v_1 \cdots v_p \mapsto v_1 \cdots v_p \otimes 2\)

so we're saying: \(H^{0-p, q}(X) \cong H^{p, q}(X^r)\)

This is rather unusual: in dim 3, it implies:

\[ H^3(X, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}(X) \cong H^0 \oplus H^1 \oplus H^2 \oplus H^3(X^r) \]
Moreover, correlation functions give extra structure on these spaces. "Yukawa couplings" on $H^3,1$ and on $H^3(X,TX)$.

On $H^3,1(X): \langle \omega_1, \omega_2, \omega_3 \rangle := \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum \eta_\beta \int_{H^3,1(X)} S_\beta \omega_1 S_\beta \omega_2 S_\beta \omega_3 e^{2\pi i \beta \omega} \over 1 - e^{2\pi i \beta \omega}$

where $\eta_\beta = "number~of~genus~0~complex~curves~in~X~representing~the~homology~class~\beta"$.

What does that mean? (e.g., can be $0$?)

In fact: Gromov-Witten invariant

We'll also see... even though it seems to involve $J$, actually $\eta_\beta$ only depends on $\omega$!

On $H^3(X,TX) \cong H^{2,1}(X): \langle \theta_1, \theta_2, \theta_3 \rangle := \int_X \omega \wedge (\theta_1 \wedge \theta_2 \wedge \theta_3)$

where $H^q(X,TX) \otimes H^p(X,\mathbb{R}) \to H^3(X, \Lambda^3 T_X \otimes \mathbb{R}_X) = H^3(X, \Omega_X) = H^{0,3}(X)$

$\theta_1 \otimes \theta_2 \otimes \theta_3 \wedge \omega$,

(0,1)-forms in $T_X$,

(3,0)-form,

(3,3)-form w/ values in $\Lambda^3 T_X$

or, in fact, thinking of $\theta_i$ as deformations of $\omega$ etc. structure $J$,

$\omega$ changes as we deform $J \sim \langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \omega \wedge \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \omega$

Gauss-Manin connection on $H^*(X,\mathbb{C})$

Need of course to assume $\omega$ normalized for this to be ok.

Mirror symmetry prediction: if $X$ & $X^\vee$ are mirror, then

$\langle \ldots \rangle$ on $H^{3,1}(X) \cong H^3(X, T_X)$, $\langle \ldots \rangle$

match under a certain change of coordinates.
More precisely, relation between $u^c$ on $X$ and $v^c$ on $X^c$ should yield a local diffeo between moduli spaces $\mathcal{M}_{\text{sym}}(X)$, $\mathcal{M}_c(X^c)$ (the "mirror map"). At the level of tangent spaces, mirror map induces $T_u \mathcal{M}_{\text{sym}}(X) = H^{1,1}(X) = H^4(X^c, TX^c) = T_{v^c} \mathcal{M}_c(X^c)$.

Then $\langle \cdot, \cdot \rangle$ match under this isomorphism.

- First concrete null prediction made using mirror symmetry:
  (Candelas-de la Ossa-Green-Parkes, 1991)
  
  $X =$ quintic Calabi-Yau 3-fold $; \deg 5$ hypersurface in $\mathbb{P}^4$

  $\implies$ predicted $n_4 = \#$ degree rational curves in $X$

  e.g.: $n_4 = 2875$ lines on the quintic (already known)
  $n_2 = 609250$

  Value for large $d$ were previously unknown

Method:

- $X^c =$ resolution of sing. of quintic for a quintic by $(2/5)^3$

- calculated Yukawa coupling on $H^4(X^c, TX^c)$

- Identified mirror map

- deduced Yukawa coupling on $H^{4,2}(X)$ & hence $n_d$'s.

Goal for 1st month of course:

- Calabi-Yau 3-folds, Complex deform. theory
- review of Hodge theory
- periods, curves, GW invariants
- understanding the statement of mirror symmetry
- example: the quintic 3-fold. [skipping lots of details]

Then we'll move on to more recent developments & other interpretations of mirror symmetry.
2. Homological mirror symmetry (Kontsevich '94)

Open string propagation = worldsheet is a surface w/ boundary
Constraints on values of fields at boundary = "D-branes"
Field theory axioms => D-branes form a category

In A-model, branes are Lagrangian submanifolds + flat bundles
In B-model, branes are complex analytic submanifolds + holom. bundle

Kontsevich gave a mathematically precise formulation:

**HMS Conj:** if \((X, J, \omega^e)\) and \((X', J', \omega'^e)\) are mirror

then\[
\begin{align*}
\text{D}^b\text{Fuk}(X, \omega^e) & \cong \text{D}^b\text{Coh}(X', J') \\
\text{D}^b\text{Coh}(X, J) & \cong \text{D}^b\text{Fuk}(X', \omega'^e)
\end{align*}
\]
equivalence between triangulated categories

→ Fukaya cat.: objects = Lagrangian submanifolds + 
 morphisms = intersection theory
  algebraic structure from Floer homology

→ coherent sheaves: objects = coherent sheaves
  morphisms = hom's & ext's of sheaves
  (think of as intersection theory!)

**NB:** ex. of coherent sheaf includes: \((Y \subseteq X \text{ subvariety} + E \to Y \text{ holom. rank bundle})\)

derived category: algebraic enlargement
  (complexes of objects, up to homotopy, ...)

→ triangulated category, i.e. 3 notion of exact triangle

This puts a healthy dose of homological algebra into mirror symmetry...
which we'll largely ignore to focus on geometry.

**Relation to previ.** Hochschild cohomology
\[
\begin{align*}
\mathcal{O}H^*(X) & \to \text{HH}^*(\text{D}^b\text{Fuk}(X)) \\
H^*(X, \Omega Tx) & \cong \text{HH}^*(\text{D}^b\text{Coh}(X))
\end{align*}
\]
Learn about:
- Lagrangian submanifolds, Floer homology,
  a simplified, naive version of Fukaya category
- sheaf cohomology, derived category
- example: the elliptic curve (after Polishchuk Zaslow)


What is it geometrically that makes manifolds mirror to each other?

Syz conj: \( X, Y \) mirrors \( \Rightarrow \) they carry mutually dual

(simplified version) special Lagrangian torus fibrations \( T^n \rightarrow X \xrightarrow{\text{L}} \mathbb{C} \)

- \( L^n = (X, \omega, \mathcal{T}, R) \) is special Lag.
  if \(\{\}
  \begin{align*}
  \omega_{1L} &= 0 \\
  \text{Im } R_{1L} &= 0 \\
  \text{then } \text{Re } R_{1L} &= \text{vol}_{1L} \text{ calibration}
  \end{align*}\)

- dual torus: \( T = \text{Hom}(\pi_1 T, \mathbb{C}^*) \) but \( U(1) \)-equivariant on \( T \)

Motivation from HMS: \( p \in X \) point \( \Rightarrow \mathcal{O}_p \in D^{\text{b}}\text{Coh}(X) \leftrightarrow L_p \in D^{\text{b}}\text{Hol}(X) \)

\[ \text{Ext}^n(\mathcal{O}_p, \mathcal{O}_p) = H^n(\mathbb{T}^n, \mathbb{C}) \text{ so } L_p = \text{Lagr. torus + } U(1)-\text{equivariance} \]

The conjecture doesn't quite hold as stated... \( \{ \text{near large complex limit} \rightarrow \text{instanton corrections} \} \)

We'll see:
- special Lagrangians and their deformations
- Lag. fibrations, affine geometry
- Ex: elliptic curve
- instanton corrections ... more example (\( k3 \), ...)

Finally: 4. extension to non-Calabi-Yau setting: Landau-Ginzburg models and their geometry; mirror symmetry for Fano varieties;
example: toric varieties.