Math 242 – Homework 2 – due Tuesday October 5, 2010.

1. a) Let M be a contact manifold equipped with a contact form α, and recall that the symplectization of (M,α) is $(\mathbb{R}\times M, d(e^r\alpha))$. Show that, for any $f\in C^\infty(M,\mathbb{R}_+)$, the symplectizations of (M,α) and $(M,f\alpha)$ are symplectomorphic. (In other terms, the symplectization only depends on the contact structure, not on the contact form).

b) Let (M,α) be a contact manifold, and let R be the Reeb vector field on M (defined by $\alpha(R)=1$, $d\alpha(R,\cdot)=0$). Consider the Hamiltonian function $H=-e^r$ on the symplectization of (M,α). Show that the associated Hamiltonian vector field is R (or more precisely, $X_H(r,x)=(0,R(x))\in T_{(r,x)}(\mathbb{R}\times M)=T_r\mathbb{R}\oplus T_xM$).

2. a) Let (M,ω) be an exact symplectic manifold, and let X be a *Liouville vector field*, i.e. a vector field such that $L_X\omega=\omega$. Let N be a hypersurface in M, such that X is transverse to TN at every point of N. Show that $\alpha=i_X\omega$ is a contact form on N.

b) Assume that the flow ϕ_t of X is defined for all $t\in\mathbb{R}$. Consider the map $\Phi:\mathbb{R}\times N\to M$ defined by $\Phi(r,x)=\phi_r(x)$. Show that $\Phi^*\omega=d(e^r\alpha)$.

(In particular, Φ induces a symplectomorphism from a neighborhood of $\{0\}\times N$ in $(\mathbb{R}\times N, d(e^r\alpha))$ to a neighborhood of N in M.)

c) Let $N\subset\mathbb{R}^{2n}$ be a star-shaped hypersurface, i.e. the image of a map $i:S^{2n-1}\to\mathbb{R}^{2n}$ of the form $x\mapsto f(x)\,x$, where $f\in C^\infty(S^{2n-1},\mathbb{R}_+)$. Let $\alpha=\frac{1}{2}\sum x_i\,dy_i-y_i\,dx_i$. Show that α is a contact form on N, and that the symplectization of (N,α) is symplectomorphic to $\mathbb{R}^{2n}\setminus\{0\}$ equipped with the standard symplectic form ω_0.

Hint: consider a suitable Liouville vector field on $(\mathbb{R}^{2n},\omega_0)$.

3. Let (V,Ω) be a symplectic vector space of dimension $2n$, and let $J:V\to V$, $J^2=-\mathrm{Id}$ be a complex structure on V.

a) Prove that, if J is Ω-compatible and L is a Lagrangian subspace of (V,Ω), then JL is also Lagrangian and $JL=L^\perp$, where $L^\perp$ is the orthogonal to L with respect to the positive inner product $g(u,v)=\Omega(u,Jv)$.

b) Deduce that J is Ω-compatible if and only if there exists a symplectic basis for V of the form

$$e_1,e_2,\dots,e_n,f_1=Je_1,f_2=Je_2,\dots,f_n=Je_n,$$

with $\Omega(e_i,e_j)=\Omega(f_i,f_j)=0$ and $\Omega(e_i,f_j)=\delta_{ij}$.

4. Show that the sphere S^6 carries a natural almost-complex structure, induced by a vector cross-product on $\mathbb{R}^7$.

Hint: view $\mathbb{R}^7$ as the space of imaginary octonions. Octonions are the non-commutative, non-associative normed division algebra structure on $\mathbb{R}^8=\mathbb{H}\oplus e\mathbb{H}$ with product given by

$$(a+be)(a'+b'e)=(aa'-\overline{b'}b)+(b'a+b\overline{a'})e,\qquad \forall a,b,a',b'\in\mathbb{H}$$

($\overline{a'}$ is the conjugate of a', i.e. $\overline{x+yi+zj+tk}=x-yi-zj-tk$). The cross-product is then defined by $u\times v=\mathrm{Im}(uv)$. (How is $\mathrm{Re}(uv)$ related to the Euclidean scalar product $\langle u,v\rangle$?) You may use without proof the fact that $\|(a+be)(a'+b'e)\|=\|a+be\|\,\|a'+b'e\|$, where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^8$. (See also Example 4.4 in McDuff-Salamon; however the method outlined there is not necessarily the simplest.)