Math 242 – Homework 1 – due Thursday September 16, 2010.

1. Show that, if E is a Lagrangian subspace of a symplectic vector space (V,Ω), then any basis $e_1,\dots,e_n$ of E can be extended to a standard basis $e_1,\dots,e_n,f_1,\dots,f_n$ of (V,Ω).

2. For which values of n does the sphere $S^{2n}\subset\mathbb{R}^{2n+1}$ carry a symplectic structure? What about the torus $T^{2n}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}=(S^1)^{2n}$?

3. Let $\{\rho_t\}_{t\in[0,1]}$ be the isotopy generated by a time-dependent *symplectic* vector field X_t on a symplectic manifold (M,ω) Then the *flux* of $\{\rho_t\}$ is defined to be

$$\mathrm{Flux}(\rho_t)=\int_0^1[i_{X_t}\omega]\,dt\in H^1(M,\mathbb{R}).$$

a) Let $\gamma:S^1\to M$ be an arbitrary closed loop, and define $\Gamma:[0,1]\times S^1\to M$ by the formula $\Gamma(t,s)=\rho_t(\gamma(s))$, so $\gamma_t(\cdot)=\Gamma(t,\cdot)$ is the image of the loop γ by ρ_t. Prove that

$$\langle\mathrm{Flux}(\rho_t),[\gamma]\rangle=\iint_{[0,1]\times S^1}\Gamma^*\omega. \tag{1}$$

(Remark: the right-hand side is simply the symplectic area swept by the family of loops $\{\gamma_t\}_{t\in[0,1]}$. In particular, equation (1) implies that this area depends only on the homology class represented by γ!)

b) Does the symplectomorphism $\phi:(x,\xi)\mapsto(x,\xi+1)$ of $T^*S^1\simeq S^1\times\mathbb{R}$ belong to the group of Hamiltonian diffeomorphisms?

Hint: assume ϕ is generated by a Hamiltonian isotopy, and use the exactness property ($\omega=d\alpha$) to rewrite the right-hand side of equation (1) in terms of the 1-form α.

4. Let N be a coisotropic submanifold in a symplectic manifold (M,ω). (i.e., at every point $p\in N$, $(T_pN)^\omega\subseteq T_pN$).

a) Show that, if X and Y are two vector fields on N such that $X,Y\in(TN)^\omega$ everywhere, then their Lie bracket $[X,Y]$ also lies in $(TN)^\omega$.

(Hint: show that, for any vector field Z on N, $\omega(Z,[X,Y])=0$, by expressing $d\omega(X,Y,Z)$ as a sum of terms including this one).

By the Frobenius integrability theorem, this implies that $(TN)^\omega$ defines an *integrable foliation* on N: given any $p\in N$, there exists a neighborhood U and a submanifold $F\subset U$, with $p\in F$, such that $TF=(TN)^\omega$ at every point of F. It is easy to check that F is an isotropic submanifold ($TF=(TN)^\omega\subset TN=(TF)^\omega$), called the *isotropic leaf* through p.

b) Assume that the isotropic foliation of N is regular, i.e. there exists a locally trivial fibration $\pi:N\to Q$ whose fibers are the isotropic leaves (connected). Show that Q carries a natural symplectic form $\tilde\omega$ such that $\omega_{|N}=\pi^*\tilde\omega$.

Hint: first show that, if W is a coisotropic subspace in a symplectic vector space (V,Ω), then Ω induces a natural symplectic structure on the quotient W/W^Ω.

5. Let (M,ω) be a symplectic manifold. The *Poisson bracket* of two smooth functions $f,g \in C^\infty(M,\mathbb{R})$ is defined by $\{f,g\} = \omega(X_f, X_g)$, where X_f and X_g are the Hamiltonian vector fields defined by f and g.

a) Show that $L_{X_g} f = \{f,g\}$, and that $[X_g, X_f] = X_{\{f,g\}}$.

Hint: to prove the second identity, you may use without proof the identity $i_{[X,Y]}\alpha = d i_X i_Y \alpha + i_X d i_Y \alpha - i_Y d i_X \alpha - i_Y i_X d\alpha$.

b) Deduce that $(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ is a Lie algebra, i.e. $\{f,g\} = -\{g,f\}$ (skew-symmetry) and $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity).

(Note: by the first identity proved in (a) and skew-symmetry of the bracket, $\{f,g\} = 0 \Leftrightarrow$ the flow of X_f preserves the level sets of g $\Leftrightarrow$ the flow of X_g preserves the level sets of f)

c) Assume that $f_1, \ldots, f_k$ satisfy $\{f_i, f_j\} = 0$ $\forall i,j$, and let $F = (f_1, \ldots, f_k) : M \to \mathbb{R}^k$. Show that any regular level set of F is a coisotropic submanifold of M, and that the vector fields X_{f_i} are all tangent to this submanifold and span the tangent space to its isotropic foliation.

(For example, if $k = \frac{1}{2}\dim M$ then the regular levels of F are Lagrangian; this situation is called an *integrable system*).