Math 215B Spring 2012 – Homework 1 – due Friday Feb 10, 2012

Hatcher 4.1 Problem 3:

For an H-space X with multiplication $\mu : X \times X \to X$ and identity element e, show that the group operation in $\pi_n(X, e)$ can also be defined by the rule $(f + g)(x) = \mu(f(x), g(x))$.

Recall that X is an *H-space* if there is a continuous multiplication map $\mu : X \times X \to X$ and an identity element $e \in X$, with $\mu(e, e) = e$, such that the two maps $X \to X$ given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to identity through maps $(X, e) \to (X, e)$.

(Examples of H-spaces include Lie groups and loop spaces.)

Hatcher 4.1 Problem 11:

Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \dots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. (An example is the infinite sphere S^∞, or more generally the infinite suspension $S^\infty X$ of any CW-complex X.)

Hatcher 4.1 Problem 15:

Show that every map $f : S^n \to S^n$ is homotopic to a multiple of the identity map by the following steps:

(a) Use Lemma 4.10 (or simplicial approximation Theorem 2C.1) to reduce to the case that there exists a point $q \in S^n$ with $f^{-1}(q) = \{p_1, \dots, p_k\}$ and f is an invertible linear map near each p_i.

(b) For f as in (a), consider the composition gf where $g : S^n \to S^n$ collapses the complement of a small ball about q to the basepoint. Use this to show that f is homotopic to a sum of maps $S^n \to S^n$ represented by invertible linear maps.

(c) Finish the argument by showing that an invertible $n \times n$ matrix can be joined by a path of such matrices to either the identity matrix or the matrix of a reflection. (Use Gaussian elimination, for example.)

Addendum: (d) Show that $\pi_n(S^n) \simeq \mathbb{Z}$. (Use homology.)