Shukel-Whitney classes

These are characterized by axiomatic properties— the existence & uniqueness will be seen later.

**Axioms**

For each real vector bundle \( \pi : E \to B \) of rank \( r \), \( \exists \) cohomology classes \( w_i(E) \in H^i(B, \mathbb{Z}_2) \), \( i = 0 \ldots r \), with \( w_0(E) = 1 \in H^0(B, \mathbb{Z}_2) \), st.

1) **Naturality:** \( E' \xrightarrow{f'} E \xrightarrow{f} B \) \( \Rightarrow w_i(E') = f^* w_i(E) \)

2) **Whitney sum:** \( E \oplus F \to B \) \( \Rightarrow w_k(E \oplus F) = \sum_{i=0}^k w_i(E) \cup w_{k-i}(F) \)

eg: \( w_1(E \oplus F) = w_1(E) + w_1(F) \)
\( w_2 = w_2(E) + w_1(E) \cup w_1(F) + w_2(F) \)

3) **Naturality:** for the tautological line bundle \( T \to B \mathbb{P}^1 \), \( w_i(T) \neq 0 \).

**Immediate properties:**

- \( E \cong E' \) isomorphic \( \Rightarrow w_i(E) = w_i(E') \).
- \( E \to B \) trivial \( \Rightarrow w_i(E) = 0 \ \forall i > 0 \) (since can write \( E \) as pullback \( B \to pt \)).
- \( E \) trivial \( \Rightarrow w_i(E \oplus E') = w_i(E') \).
- \( E \) Eucl. rank \( r \), with \( k \) pairwise linearly indep. sections \( \Rightarrow w_i(E) = 0 \) for \( i > r - k \).

For convenience, def. \( H^*(B, \mathbb{Z}_2) := \prod_{i=0}^\infty H^i(B, \mathbb{Z}_2) \) total cohomology ring

\[ \exists a_0 + a_1 + a_2 + \ldots, a_i \in H^i(B, \mathbb{Z}_2) \]

\( \cup \) define a graded commutative ring structure.

and let the total Shukel-Whitney class \( W(E) := 1 + w_1(E) + \ldots + w_r(E) \in H^*(B, \mathbb{Z}_2) \).

Then \( W(E \oplus F) = W(E) \cup W(F) \)

**Lemma:** Elements of the form \( a = 1 + a_1 + a_2 + \ldots \in H^*(B, \mathbb{Z}_2) \) form a group under mult.

**Pf:** \( a^{-1} = 1 + (-a_1) + (-a_2) + \ldots \) solve inductively for \( a_k \) by looking at deg \( k \) part of \( a \cdot a^{-1} \).
namely \( a_k + a_{k-1} \bar{a}_1 + \cdots + a_1 \bar{a}_{k-1} + \bar{a}_k = 0 \) determines \( \bar{a}_k \) once \( \bar{a}_1, \ldots, \bar{a}_{k-1} \) known.

(in fact \( \bar{a}_1 = a_1 \), \( \bar{a}_2 = a_1^2 + a_2 \), \( \ldots \))

Conclude: \( U(E) = U(F)^{-1} U(E \oplus F) \). In particular if \( E \oplus F \) is trivial then \( U(E) = U(F)^{-1} \).

E.g. this applies to: \( M \subset \mathbb{R}^n \) small submanifold \( \Rightarrow TM \oplus NM \cong \mathbb{R}^n |_M \) or \( M \) immersed into \( \mathbb{R}^n \) so \( U(NM) = U(TM)^{-1} \).

Example: \( S^n \subset \mathbb{R}^{n+1} \) has trivial normal bundle (vanishing scalar section \( s(k) = k \))

so \( U(TS^n) = U(NS^n)^{-1} = 1 \).

Skylit. Whitney claim doesn't detect the nontriviality of \( TS^2 \).

[Note: \( TS^3 \) is trivial! \( T S^3 \cong \mathbb{R} \) \( \Rightarrow T_0 S^3 \cong 0 \).]

Recall: the cohomology of \( \mathbb{R}^n \) is \( H^i(\mathbb{R}^n, \mathbb{Z}_2) = \mathbb{Z}_2 \) \( \forall 0 \leq i \leq n \).

(1) Cellular chain complex \( C_i = \mathbb{Z}_2 \xrightarrow{d} C_{i-1} = \mathbb{Z}_2 \)

and \( \text{Hom}(\; , \mathbb{Z}_2) \) becomes \( C^i = \mathbb{Z}_2 \xrightarrow{d^*} C^{i+1} = \mathbb{Z}_2 \).

and as ring, denoting by \( a \in H^i(\mathbb{R}^n, \mathbb{Z}_2) \) the nonzero line

\( a^k \) a generator of \( H^k(\mathbb{R}^n, \mathbb{Z}_2) \), \( \forall k \), i.e. \( H^k(\mathbb{R}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2 \langle a \rangle / a^{k+1} \).

(Prove by induction on \( n \). If true for \( \mathbb{R}^n \) then:

incl. \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \) induces \( a \mapsto a^* \) on \( H^0, \ldots, H^n \), maps \( a \mapsto a \), and preserves multiplicative structure.

so true up to \( a^n \). Moreover Poincaré duality (on \( \mathbb{Z}_2 \)):

\( H^i(\mathbb{R}^{n+1}, \mathbb{Z}_2) \oplus H^n(\mathbb{R}^{n+1}, \mathbb{Z}_2) \cong H^{i+1} = \mathbb{Z}_2 \) implies

\( a^n \cdot a = a^{n+1} \neq 0 \).

Example 2: \( \mathcal{T} \rightarrow \mathbb{R}^n \) tautological bundle \( \Rightarrow U(\mathcal{T}) = 1 + a \).

If pullback by inclusion \( \mathcal{R}^n \subset \mathbb{R}^{n+1} \) taut. bundle of \( \mathcal{R}^n \)

so \( i^* \mathcal{T} = \mathcal{R}^n \) \( \Rightarrow \mathcal{T} \mid \mathcal{R}^n = \text{taut. bundle of } \mathcal{R}^n \).

\( \Rightarrow i^* \mathcal{T} \cdot a = a^* \mathcal{T} \cdot a = a^{n+1} \neq 0 \).

Example 3: tangent bundle of \( \mathbb{R}^n \),

\( \text{Lemma. } T \mathbb{R}^n = \text{Hom}(\mathcal{T}, \mathcal{T}^\perp) \) (\( \mathcal{T}^\perp \) other coframe of \( \mathcal{T} \subset \mathbb{R}^{n+1} \)).
\[ TS^n = \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^{n+1} | x \cdot v = 0\} \]

The above is the same up to \( (x,v) \sim (-x,-v) \) (antipodal involution). Such a pair \((x,v) \Leftrightarrow \text{linear mapping } L : L = \mathbb{R} \cdot x \rightarrow \mathbb{R}^\perp, \quad t \mapsto tv\)

and this gives \( T^x \mathbb{R}^n \cong \text{Hom}(L, L^\perp) \) an algebra.

**Corollary:** \( T^x \mathbb{R}^n \oplus \mathbb{R} \cong \text{Hom}(\tau, \tau^\perp) \oplus \text{Hom}(\tau, \tau) \)

\[ = \text{Hom}(\tau, \tau^{n+1}) = \tau^k \oplus \cdots \oplus \tau^0 \cong \tau \oplus \cdots \oplus \tau. \]

Hence \( w(T^x \mathbb{R}^n) = w(T^x \mathbb{R}^n \oplus \mathbb{R}) = w(\tau)^{n+1} = (1+a)^{n+1}. \)

**Example:**
- \( w(\mathbb{R}P^1) = 1 \)
- \( w(\mathbb{R}P^2) = 1 + a + a^2 \)
- \( w(\mathbb{R}P^3) = 1 \)
- \( w(\mathbb{R}P^4) = 1 + a + a^2. \)

**Corollary:** (Stiefel) \( \mathbb{R}P^n \) is parallelizable \( \Rightarrow n = 2^k - 1 \in \{1, 3, 7, 15, \ldots \} \)

[only case where \( \binom{n+1}{i} \) all even \( \forall 1 \leq i \leq n \), \[ \text{only } \mathbb{R}P^1, \mathbb{R}P^3, \mathbb{R}P^7 \text{ parallelizable using } \mathbb{C}, \mathbb{H}, \mathbb{O}; \text{ other actually aren't.} \]

**Application:** If \( M^n \) admits an immersion into \( \mathbb{R}^{n+k} \) then \( w(TM)^{n+1} = 1 + \bar{v}_1 + \bar{v}_2 + \ldots \) has \( \bar{v}_i = 0 \) for \( i > k \).

**Example:** for \( \mathbb{R}P^4 \), \( w(\mathbb{R}P^4)^{-1} = 1 + a + a^2 + a^3 \) so \( \mathbb{R}P^4 \nRightarrow \mathbb{R}^6 \)

but Whitney's Hopf \( \mathbb{R}P^4 \nRightarrow \mathbb{R}^{2^2 - 1} = \mathbb{R}^7 \); 7 is optimal.

(similarly for \( n = 2^m \), \( \mathbb{R}P^{2^m - 2} \nRightarrow \mathbb{R}^{2^m - 1} \), although Whitney is \( \mathbb{R} \leq \mathbb{R}^{2^{m-1}} \).

**Stiefel-Whitney Numbers and Cobordisms:**

- To get numerical invariants of closed manifolds, rather than cohomology classes:
  - integrate \( w_i(TM) \) against fundamental class \([M] \in H_n(M, \mathbb{Z}/2)\).

**Stiefel-Whitney numbers** := \( \langle w_1(TM), \ldots, w_n(TM), [M]\rangle \in \mathbb{Z}/2 \quad \forall i \text{ st. } \sum i = n. \)
For $n$ odd, $n = 2k - 1$, $u(T_{R^P}) = (1 + a)^{2k} = (1 + a)^k$
so all odd $u_i$s are zero $\Rightarrow$ all Stiefel-Whitney $w_i$s are zero
for $n$ even, $u_n(T_{R^P}) = (n + 1) a^n = a^n \neq 0$, and $w_i^n = ((n + 1) a)^n = a^n \neq 0$

Why we care? Let $M$ be a closed smooth $n$-manifold, not necessarily connected.

Then (Pontryagin $\Rightarrow$) \quad $\exists$ B smooth compact $(n+1)$-manifold with boundary $\partial B = M$
Then $\Rightarrow$ all the Stiefel-Whitney numbers of $M$ are zero.

Pf of $\Rightarrow$ ($\Leftarrow$ is much harder).

Let $[B] \in H_{n+1}(B, M; \mathbb{Z}_2)$ fundamental class, then $[M] = \partial([B])$ under
$\partial: H_{n+1}(B, M) \to H_n(M)$.

Note $TB$ well-defined (even at boundary) and $TB|_M = TM \oplus \mathbb{R}$

so $u_i(TB)|_M = u_i(TM)$.

Hence any polynomial $P$ in Stiefel-Whitney classes of $TM$ is in the image of
$H^n(B) \xrightarrow{i^*} H^n(M)$, hence $P \in \ker((S : H^n(M) \to H^{n+1}(B, M)))$.

Thus $\langle P, [M] \rangle = \langle P, \partial([B]) \rangle = \langle S(P), [B] \rangle = 0$.

Contrary: $M_1, M_2$ smooth closed $n$-manifolds are unorientedly smoothly cobordant
(i.e. $\exists$ smooth compact manifold $B^{n+1}$, not necessarily st.
$\partial B = M_1 \cup M_2$) iff all of their Stiefel-Whitney numbers are equal.