Fiber bundles
\[ F \xrightarrow{i} E \xrightarrow{p} B \]
where all fibers \( p^{-1}(b) \subseteq E \) are homeomorphic to each other

+ more...

ex. trivial fibration \( E = F \times B \xrightarrow{p} B \)

Essentially, a fiber bundle is a "twisted product."

eg. Möbius band = internal bundle over \( S^1 \).

These give rise to, e.g., in homotopy as via homotopy lifting property

**Def.**
- \( p : E \rightarrow B \)
  - homotopy lifting property with a space \( X \) if
    - \( \forall \) homotopy \( g : X \rightarrow B \)
    - \( \forall \) lift \( \tilde{g}_0 : X \rightarrow E \)
    - \( \exists \) homotopy \( \tilde{g}_t : X \rightarrow E \) lifting \( g_t \).
  - a fibration is a map \( p : E \rightarrow B \) s.t. homotopy lifting property holds
    - for all spaces \( X \).

**Ex.**
- for \( B = F \xrightarrow{\text{proj}} B \),
  - given \( g_t \) and \( \tilde{g}_0(x) = (g_0(x), h(x)) \),
  - take \( \tilde{g}_t(x) = (g_t(x), h(x)) \).

**Thm.**
Suppose \( p : E \rightarrow B \) has homotopy lifting property with disks \( D^k \) \( \forall k \geq 0 \).
- Choose base pts \( b_0 \in B, x_0 \in E = p^{-1}(b_0) \).
- Then \( p_* : \pi_n(E, x_0) \rightarrow \pi_n(B, b_0) \)
  - is an isom. \( \forall n \geq 1 \), and if \( B \) is path-connected we have
  - \( \cdots \rightarrow \pi_n(F, x_0) \xrightarrow{p_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, x_0) \rightarrow \cdots \rightarrow \pi_0(E, x_0) \rightarrow 0 \)

**Rmk.**
- in fact the proof will use a relative version for \( (D^k, S^k) \).
  - homotopy lifting prop. for \( X : \) given map \( X \times \{0\} \rightarrow B \)
    - and lift to \( E \) of its restriction to \( x \times \{0\} \), \( \exists \) lift on \( X \times \{1\} \).
  - For \( (X, A) \):
    - \( (X \times \{0\}) \cup (A \times \{1\}) \)

Since \( (D^k \times I, S^k \times 0) \) is homotopy eq. to \( (D^k \times I, D^k \times 0 \cup D^k \times I) \),
- homotopy lifting for \( D^k \) \( \iff \) for \( (D^k, \partial D^k) \).
- and in fact, by induction on cells, \( \cdots \) for all CW-pairs \( (X, A) \)
  - enough to extend homotopy over one cell at a time
- map w/ homotopy lifting property for disc = "Serre fibration"
- as enough for all practical purposes.
pf. Thm: 1) show $p_\varepsilon$ is onto: given $[f] \in \pi_n(B, b_0)$ map by $f: (I^n, \partial I^n) \to (B, b_0)$

lift $f$ to contact map to $x_0$ over $J_n = (I^{n-1} \times \{0\}) \cup (\partial I^n \times I) \subset \partial I^n$

Then by homotopy ext. prop for $(I^n, \partial I^n)$

this extends to a lift $\tilde{f}: I^n \to E$ of $f$.

Moreover, $\tilde{f}(\partial I^n) \subseteq F$ since lifts $f(\partial I^n) = b_0$.

Hence $\tilde{f}$ represents an element of $\pi_n(E, F, x_0)$

and $p_\varepsilon [\tilde{f}] = [p_\varepsilon \tilde{f}] = [f]$.

2) injectivity: given $\tilde{f}_0, \tilde{f}_1: (I^n, \partial I^n, J_n) \to (E, F, x_0)$ s.t.

$p_\varepsilon [\tilde{f}_0] = p_\varepsilon [\tilde{f}_1]$, let $G: (I^n \times I, \partial I^n \times I) \to (B, b_0)$ homotopy

from $p_\varepsilon \tilde{f}_0$ to $p_\varepsilon \tilde{f}_1$. We have a partial lift by $\tilde{f}_0, \tilde{f}_1$, and $x_0$

of $G$ over $I^n \times \{0\} \cup I^n \times \{1\} \cup \overline{J_n} \times \{0, 1\}$

which after pointing ends of $I^n \times I$.

is another instance of relative lifting problem

for $(I^n, \partial I^n)$. Hence $\exists$ lifting $\bar{G}$ over $I^n \times I$.

By contradiction, on each $I^n \times \{t\}, \bar{G}$ maps $\overline{J_n} \times \{t\}$ to $x_0$

and $\partial I^n \times \{t\}$ to $F$ since $G(\partial I^n \times \{t\}) = b_0$.

Hence $\bar{G}$ is a homotopy of maps $(I^n, \partial I^n, J_n) \to (E, F, x_0)$

and $[\tilde{f}_0] = [\tilde{f}_1]$.

The l.e.s. then follows from l.e.s. in rel. homotopy:

$\pi_n(F, x_0) \xrightarrow{\partial_\varepsilon} \pi_n(E, x_0) \xrightarrow{\partial_\varepsilon} \pi_{n-1}(E, F, x_0) \xrightarrow{\partial_\varepsilon} \pi_{n-1}(F, x_0) \to \ldots$

$\pi_n(B, b_0)$

$\simeq \pi_1(B, b_0)$

and subjectivity of $\pi_0(F) \to \pi_0(E)$ at the end follows from

path connectedness of $B$ and homotopy lifting property: lift path from

any $b \in B$ to $b_0$ to a path from any $x \in E$ to a point of $F$.

In practice, nicest class of fibrations = fiber bundle
**Def:** Fiber bundle structure on $E$ with fiber $F :=$ projection map $p:E \to B$

- each point of $B$ has a wld $U$ st. $\exists$ homeomorphism $h_i:p^{-1}(U) \to U \times F$
- diagram

\[
\begin{array}{c}
\xymatrix{
p^{-1}(U) \ar[r]^{h} \ar[d] & U \times F \ar[d] & \text{is commutative} \\
p(U) \ar@/^1pc/[u] & \text{proj.} \ar@/_1pc/[u]
}\end{array}
\]

**Rmk:** hence $h$ identifies each fiber $F_b = p^{-1}(b) \approx F$ for $b \in U$.

Such $h$ is called a **local trivialization** of the bundle.

**Rmk:** the difference $h_1$ in two local trivializations $h_1$ on $U_1$, $h_2$ on $U_2$

\[
(U_2 \cap U_1 \neq \emptyset):
\begin{align*}
(h_1 \circ h_2^{-1})(b, f) &= (b, (\varphi_{12}(b, f))
\end{align*}
\]

i.e. $\varphi_{12} : U_{12} \to \mathrm{Homeo}(F) \quad \text{"change of trivialization"}

Later we'll look at bundles with extra structure, where $\varphi_{12}$ is constrained
to lie in a specific subgroup of homeos; most notably **vector bundles**, with fiber $\approx$ a vector space $V$, and we fix an equivalence class of local
tivializations $\varphi_{ij}$: transition functions $\varphi_{ij}$ take values in $\mathrm{GL}(V)$ (linear automorphisms).

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**Ex:** a fiber bundle w/ fiber a discrete space is a covering space.

Conversely, a covering space over a connected base (so $\vert \text{fiber} \vert = \text{constant}$)
is a fiber bundle.

- **Dobinski band** $\mathbb{I} = [-1, 1] / (0,0) \sim (1, -1)$

\[
\begin{array}{c}
\xymatrix{
-1 \ar[r] & 1 & \text{fiber bundle w/ fiber } [-1, 1] \\
\mathbb{I} / 0 & = S^1
\end{array}
\]

- **projective space**: complex analogue of double over $S^n \to \mathbb{R}P^n$ is a fiber bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$.

\[
\begin{align*}
\text{unit sphere in } \mathbb{C}^{n+1} & = S^{2n+1} \\
\text{set of lines in } \mathbb{C}^{n+1} & \sim (\mathbb{C}^{2n} \times \mathbb{C}^{2n}) \setminus \{0\} \forall \lambda \in S^1
\end{align*}
\]

To see local triviality: over $U_i = \{(z_0, \ldots, z_n) / z_i \neq 0\}$, $p^{-1}(U_i) \approx U_i \times S^1$

\[
(z_0, \ldots, z_n) \mapsto ([z_0, \ldots, z_n], z_i / |z_i|)
\]
In fact, we can glue together a fiber bundle $S^1 \to S^0 \to \mathbb{CP}^\infty$.

Particularly interesting is the Hopf bundle $S^1 \to S^3 \to S^2 (= \mathbb{CP}^1)$

\[ (z_1, z_2) \mapsto \frac{z_2}{z_1} \in S^1 \]

Rule: points of circles\( \circ \) are tori \( |z_1|^2 = \frac{1}{1+z_2^2}, \quad |z_2|^2 = \frac{r}{1+z_2^2} \) \( \quad S^1 \times S^1 \)

\( \quad S^3 \equiv \mathbb{CP}^1 \)

On each torus, fibers\( \circ \)

(Defining number of fibers w/ each other is \( 1 \) !)

Over quaternions: \( S^2 \to \mathbb{S}^{n+3} \to \mathbb{HP}^n \), e.g. \( S^3 \to S^7 \to S^4 \)

(and for octonions \( S^{7} \to S^{15} \to S^{8} \); doesn't really extend \(-\mathbb{OP}^n\). D not add. !)

\( (z_0 \cdots z_{n}) \times (z_0 \cdots z_{n}) \) not equiv addition \( \equiv \mathbb{OP}^2 \) through.

\( \mathbb{OP}^2 = S^0 \cup \mathbb{E}^8 \) attach by Hopf map.

Just as for \( \mathbb{CP}^2 \) \( \times \) \( \mathbb{HP}^2 \).

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Prop: Fiber bundles have the homotopy lifting property w/ all CW-pairs

(\( \text{in fact, all spaces if base is paracompact} \))

We don't care.

Let \( p: E \to B \) fiber bundle w/ fiber \( F \),

\( G: I^n \times I \to B \) a homotopy to lift, starting w/ \( g_0 \) lift of \( g_0 \) to \( F \).

\( G(x, t) = g_t(x) \)

Choose open cover \( \{ U_\alpha \} \) of \( B \) w/ local homotopies \( h_\alpha: p^{-1}(U_\alpha) \to U_\alpha \times F \).

Compactness of \( I^n \times I \) \( \Rightarrow \) can subdivide \( I^n \) into small cubes \( C_i \) into small intervals \( I_j = [t_j, t_{j+1}] \)

\( \Rightarrow \) \( G \) maps each \( C_i \times I_j \) to a single \( U_\alpha \).

By induction on \( n \), can assume \( g_\alpha \) has already been constructed over \( \partial C_i \times V_i \).

To extend over \( C_i \), proceed in stages over each time interval \( I_j \).

Thus can assume \( G: I^n \times I \to U_\alpha \), and given \( G: I^n \times 0 \to U_\alpha \times F \).

However, product case \( \Rightarrow \) homotopy lifting holds! Namely:

\( \tilde{G}(x) = (g_\alpha(x), h_\alpha(t)) \mapsto F \), already given on \( J \); extend using \( I^n \times I \to J \times F \).
The fibre bundle gives $\pi_3$ is homotopy.

**Ex.** Given space $p: E \to B$, $E \& B$ path-connected ($F$ discrete):

$$\pi_n(E) \xrightarrow{p_0} \pi_n(B) \quad \forall n \geq 2 \quad (\pi_n(F), \pi_n(\partial F) = 0).$$

and $0 \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to 0$ (already known!)

- $S^1 \to S^0 \to \mathbb{C}P^\infty$ give $\pi_n(\mathbb{C}P^\infty) \simeq \pi_{n-1}(S^1)$ so $\mathbb{C}P^\infty$ is $K(\mathbb{Z}, 2)$.

- $S^1 \to S^3 \to S^2$ give $\pi_2(S^3) \cong \pi_1(S^1) = \mathbb{Z}$

and for $n \geq 3$, $\pi_n(S^3) \xrightarrow{p_0} \pi_n(S^2)$, in particular $\pi_3(S^2) = \mathbb{Z}$.

(Cog: $S^2$ and $S^3 \times \mathbb{C}P^\infty$ are $1$-connected have same homotopy groups, but of course not homology equivalent.)

**Ex:** Whitehead products. Let's compute $\pi_3(S^2 \vee S^2)$. (From general in book)

Consider $S^2 \vee S^2 \to S^2 \times S^2 \simeq (S^2 \vee S^2) \vee 4$-cell.

Observe $\forall n$, $\pi_n(S^2 \vee S^2) \to \pi_n(S^2 \times S^2) \cong \pi_n(S^2) \oplus \pi_n(S^2)$ is surjective.

So let $s$ be the pair splits into s.e.s.

$$0 \to \pi_{n+1}(S^2 \times S^2 \vee S^2 \vee S^2) \to \pi_n(S^2 \vee S^2) \to \pi_n(S^2 \times S^2) \to 0$$

For $n = 3$:

$$0 \to \mathbb{Z} \to \pi_3(S^2 \vee S^2) \to \mathbb{Z}^2 \to 0.$$

by Hurewicz, gen'd by the 4-cell.

or by exact pair $\pi_4(S^2 \times S^2, S^2 \vee S^2) = \pi_4(S^2 \times S^2 / S^2 \vee S^2) = \pi_4(S^4) = \mathbb{Z}$.

hence $\pi_3(S^2 \vee S^2) \cong \mathbb{Z}^2$, generated by:

- $\text{Map}$ maps $S^3 \to S^2$.
- Attaching map of 4-cell of $S^2 \times S^2$

This is a special case of the Whitehead product $\pi_k(X) \times \pi_1(X) \to \pi_{k+1}(X)$

**Ex:** $S^k \to X$, let $[f, g] = \text{attaching map}$ of $k$ cell in $S^k \times S^k$.

In our case: the nontrivial gen of $\pi_3(S^2 \vee S^2)$ is $[i_1, i_2]$ and $i_1, i_2 \in \pi_2(S^2 \vee S^2)$ inc'd of $S^2$ factors.
Ex.: $S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1}$ generalizes to: (case $k=1$ of)

$$U(k) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$$

fiber bundle

unitary grp $\to \mathbb{C}^n$
Grassmannian of $k$-planes in $\mathbb{C}^n$

Stiefel refld :=

space of $k$-frames, i.e.
$\langle v_i, \ldots, v_k \rangle$ orthonormal vectors in $\mathbb{C}^n \mapsto \text{span}(v_i, \ldots, v_k) \subset \mathbb{C}^n$

space of unitary basis of a given $k$-dim subspace $\approx U(k)$

Similarly over $\mathbb{R}$ with $O(k)$ fiber, $\mathbb{H}$ with $\text{Sp}(k)$ fiber.

• For $k=1$, $V_1$ = sphere is highly non-$\sigma$; generally:

$$\begin{align*}
V_k(\mathbb{R}^n) &\approx (n-k-1)\text{-ann.} \\
V_k(\mathbb{C}^n) &\approx (2n-2k) \\
V_k(\mathbb{H}^n) &\approx (4n-4k+2)
\end{align*}$$

and $V_k(\mathbb{R}^0) \approx \text{constant}.$

This is proved by induction on $k$, using

$$V_{k+1}(\mathbb{R}^{n+1}) \to V_k(\mathbb{R}^n) \to S^{n-1}$$

($p^{-1}(e_i) = \{ (k-1)\text{-frame in } e_1 \subset \mathbb{R}^{n-1} \}$)

& induction l.e.s.

(similarly $\mathbb{C}$, $\mathbb{H}$)

• Specializing to $k=n$, noting $V_n(\mathbb{R}^n) \approx O(n)$:

we get a fiber bundle $O(n-1) \to O(n) \to S^{n-1}$

$\langle e_1, \ldots, e_n \rangle \to e_1$

i.e. $A \mapsto A \langle \frac{1}{\sqrt{2}} \rangle$

Similarly $U(n-1) \to U(n) \to S^{2n-1}$

$\text{Sp}(n-1) \to \text{Sp}(n) \to S^{4n-1}$

Hence homotopy groups of $O(n)$, $U(n)$, $\text{Sp}(n)$ related to those of sphere!

Corollary: the inclusion $O(n-1) \subset O(n)$ induces an isom. on $\pi_i$ for $i \leq n-3$.

Hence $\pi_i O(n)$ indept. of $n$ for $n \gg 1$.

Similarly for $U(n)$, $\text{Sp}(n)$. Surprising fact: these limit groups have a simple structure!
But periodicity holds: for \( n \gg i \), \( \pi_i U(n) = \begin{cases} 0 & i = 0 \mod 2 \\ \mathbb{Z} & i = 1 \mod 2 \end{cases} \) and \( 2 \cdot \text{periodic} \)

\( \pi_i O(n) = \begin{cases} \mathbb{Z}_2 & i = 0, 1 \mod 8 \\ \mathbb{Z} & i = 3, 7 \mod 8 \end{cases} \) \& periodic

\( \pi_i \mathfrak{sp}(n) = \pi_{i+8} O(n) \).

Come from an argument showing \( SU(\infty) \simeq \mathbb{Z} \times G_{\infty}(C^\infty) \)

\( \nu \in G_{\infty}(C^\infty) \)

\( \implies \pi_i; SU(\infty) \simeq \pi_i; (SU(\infty)) \simeq \pi_i; (G_{\infty}(C^\infty)) \simeq \pi_{i+8} (U(\infty)) \)

les., \( U(\infty)\text{ contravariant} \)

- The stabilization properties of \( \pi_i \) of lie groups are remarkably simple.

For other spaces: \( \text{Fredholm} \Rightarrow \pi_i (X) \rightarrow \pi_{i+1} (SX) \rightarrow \pi_{i+2} (S^2 X) \rightarrow \ldots \)

\( \text{iso if } X \text{ is } i \cdot \text{connected} \text{... iso if } SX \text{ is } (i+1) \cdot \text{connected} \text{...} \)

\( \text{are eventually isom.} \)

\( \Rightarrow \text{stable homotopy groups } \pi_i^S (X) = \text{limit of this.} \)

For spheres: \( \pi_i^S (S^n) = \pi_{i+2n} (S^n) \) for \( n > i+1 \).

\[ \text{(Serre): } \sum \pi_i^S \mid \text{these are all finite for } i > 0. \]

First few: \( i = 0, 1, 2, 3, 4, \ldots \) (gets crazy)

\( \pi_i^S = \mathbb{Z}_2 \mathbb{Z}_2 \mathbb{Z}_2 \mathbb{Z}_2 0 \ldots \)

There's an interesting operation: composition \( S^i \times S^j \rightarrow S^{i+j} \rightarrow S^k \) \( k \gg i, j \)

induce a map \( \pi_i^S \times \pi_j^S \rightarrow \pi_{i+j}^S \)

Thus composition maps make \( \bigoplus \pi_i^S \) a graded-commutative ring:

\[ [\alpha, \beta] = (-1)^{ij} \beta \alpha \]
Also, since we’ve been discussing fiber bundles & more generally fibrations:

**Prop:** \( p: E \to B \) fibration on path-connected \( B \Rightarrow \) the fibers \( F_b = p^{-1}(b) \) are all homotopy eq., i.e., homeomorphic to a fiber bundle.

(idea: homotopy lifting prop applied to \( f: F_{b_0} \times I \to B \)
\((x, t) \mapsto \gamma(t) \) path \( b_0 \sim b, \in B \)
& given \( \delta t \), \( \tilde{F}_0 = \text{inclusion: } F_{b_0} \to E \)
HLP gives \( \tilde{f}: F_{b_0} \to F_{b_1} \) show h.e. by considering converse map \( F_{b_0} \to F_{b_0} \) homotopy to id)

**Pullback contraction:**

\[ p: E \to B \text{ fibration (rep fiber bundle), } f: A \to B \]
\[ \Rightarrow \text{ pullback fibration (rep bundle)} \]
\[ f^*E \equiv \{(a, x) \in A \times E / f(a) = p(x)\} \]
\[ \xrightarrow{\pi} A \\
\]
\[ \text{commutative diagram } f^*E \to E \]
\[ \pi \downarrow \quad \downarrow p \\
A \quad \to \quad B \]

Easy to check: homotopy lifting prop. for \( p = \pi \) as well, local hiv: for \( p = \pi \) as well.

**Prop:** \( f_0, f_1: A \to B \) homotopic \( \Rightarrow f_0^*E, f_1^*E \) are fiber-preserving homotopy equivalent.

(i.e. \( \exists \) \( \pi: \tilde{A} \to A \))

\[ f_0^*E \cong f_1^*E \]

**Corollary:** A fibration over a contractible base is fiber-preserving homotopy eq. to a product fibration \( p: F \times B \to B \).

(apply prop. to \( id^*E = E \) vs. \( r^*E = F_{b_0} \times B \) where \( r: B \to \{b_0\} \).)