Calculation methods: excision & Hurewicz

πₙ is much harder to compute than Hn because excision doesn’t work in general! However, there’s still something.

```
A ∩ C ∩ B
```

X = A ∪ B, A∩B = C

(X,B) vs. (A,C)?

Thus:

X = A ∪ B CW-complex, A,B subcomplexes, A∩B = C nowhere connected

If (A,C) is m-connected and (B,C) is n-connected then

the inclusion map induces \( \pi_i(A,C) \to \pi_i(X,B) \)

isom for \( i < m+n \)

souction for \( i = m+n \).

Note: up to \( i < n \) is not surprising since we’ve seen before that, by replacing by CW-approximations, can assume B∩C only has cells of dim \( \geq n+1 \).

Corollary: Freudenthal suspension theorem

The suspension map \( \pi_i(S^n) \to \pi_{i+1}(S^{n+1}) \) is an iso for \( i < 2n-1 \)

souction for \( i = 2n-1 \).

More generally the same holds for \( \pi_i(X) \to \pi_{i+1}(SX) \) whenever

X is an \((n-1)\)-connected CW-complex.

(Suspension map: \( S^i \to X \to S^{i+1} = S(S^i) \to SX \))

Proof: write \( SX = C_+X \cup C_-X \) interestingly along \( X \).

Then suspension map \( \pi_i(X) \to \pi_{i+1}(C_+X, X) \xrightarrow{\text{inc}} \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX) \)

isom for \( i \geq 0 \)

implying for \( i < 2n-1 \)

\( \pi_i(X) \to \pi_{i+1}(C_+X) \to \pi_{i+2}(C_+X) \to \ldots \)

\( \cong 0 \) (contractible)

ie. cone map \( S^i \to X \to C(S^i) : D^i \to CX \)

\( \cong 0 \) for \( i < 2n \)

\( \cong C(X) \) for \( i = 2n \)

\( \cong 0 \) for \( i > 2n \)

\( X \) \((n-1)\)-connected \( \Rightarrow (CX,X) \) is \( n\)-connected, so 14.1 (iso for \( i+1 < 2n \))
Corollary: \( \forall n \geq 1, \pi_n(S^n) \cong \mathbb{Z} \) gen. by identity map. [see also Thm 1].

\[ \text{pf: Fundamental \Rightarrow suspending induces } \pi_1(S^1) \to \pi_2(S^2) \cong \pi_3(S^3) \cong \ldots \]
\[ \Rightarrow \pi_n(S^n) \text{ for } n \geq 2 \text{ is cyclic (finite a finite).} \]
\[ f_k = k \text{ id is a map of degree } k, \text{ i.e. } f_k[S^n] = k[S^n] \in \pi_n(S^n) \cong \mathbb{Z} \]
\[ \text{so } f_k \text{ pairwise non-homotopic, } \pi_n(S^n) \cong \mathbb{Z}. \]

Note however \( \pi_3(S^2) = \mathbb{Z} \) (gen. by map \( S^3 \to S^2 \))
\[
\begin{array}{c}
\pi_4(S^3) = \pi_5(S^4) = \ldots = \mathbb{Z}/2.
\end{array}
\]

(\( \pi_{n+k}(S^n) \) stabilizes for \( n \geq k+2 \! \).)

\[ \text{Proof of Thm: successive case of increasing generality:} \]

- \textbf{Case 1:} - assume \( A = C \cup (m+1)-\text{cells } e^m \)
\[ B = C \cup \text{single } (n+1)-\text{cell } e^{n+1}. \]

(a) \textbf{To show surjectivity of } \( \pi_i(A,C) \rightarrow \pi_i(B,C) \) for \( i \leq n+m \):

let \( f: (I^i, \partial I^i, J^i) \rightarrow (X, B, \partial X) \) - want to push \( f \) away from \( e^{n+1} \)?
\[ \text{I compact \Rightarrow image of } f \text{ is compact, so needs only finitely many of } e_{x}^{n+1} \text{'s.} \]
\[ \text{PL approximation lemma: can homotope } f \text{ so } \exists \text{ simplex } \Delta_{x}^{m+1} \subset \text{int}(e^{m+1}) \]
\[ \Delta_{x}^{m+1} \subset \text{int}(e^{m+1}) \]
so that \( f^{-1}(\Delta_{x}^{m+1}), f^{-1}(\Delta_{x}^{n+1}) \) finite unions of convex polyhedra on which \( f \) is PL.
(on each, \( f \) is a linear map \( \mathbb{R}^i \rightarrow \mathbb{R}^{m+1} \cap \mathbb{R}^{n+1} \); can assume the linear maps are surjective (else take smaller \( \Delta_{x}^{m+1}, \Delta_{x}^{n+1} \) to avoid image of low-rank maps).

\[ \text{key observation: for } q \in \Delta_{x}^{m+1}, f^{-1}(q) = \text{finite union of convex polyhedra of} \]
\[ \text{dim } \leq i-n-1. \]
\[ \text{For } p \in \Delta_{x}^{m+1}, f^{-1}(p) = \text{polyhedron } \text{dim } \leq i-n-1. \]

These are of course mutually disjoint, but we can do better - observe
\[ i \leq m+n = (i-n-1) + (i-n-1) < i-1. \text{ So if we choose } q, p \text{ generally,} \]
the images of these polyhedra under \( \pi: I^i \rightarrow I^{i-1} \) (first last coord.) are disjoint.
(specifically, choose \( p_x \in \Delta_{x}^{m+1}, f(\pi^{-1}(\pi(f^{-1}(q)))) \) polyhedron \( \text{dim } \leq i-n \).)
Hence, Lemma:

If \( i \leq m+n \), \( \exists x \in \Delta_{\alpha}^{m+n} \), \( q \in \Delta_{\beta}^{m+n} \) and \( \varphi : I^{i-1} \to [0,1] \)

\[ \exists \; \eta \in \Delta_{\beta}^{i} \text{ s.t. } \varphi = 0 \quad \text{on} \quad \Delta_{\beta}^{i}, \quad \varphi^{-1}(q) \in \text{ below graph}(\varphi) \]

\[ \varphi^{-1}(p_{\alpha}) \in \text{ above graph}(\varphi) \quad \forall \alpha. \]

This allows us to excise the portion of \( f \) below graph(\( \varphi \)) by a homotopy:

Let \( f_t = \text{ restriction of } f \text{ to region above } \text{ graph}(\varphi) \) \( \quad 0 \leq t \leq 1 \)

(identifying it with \( I^{i-1} \))

By construction:

1. \( \forall t, \; f_t(I^{i-1}) \text{ is disjoint from } P = \bigcup \{ x \} \quad \text{for } \alpha \neq \beta \)
2. \( f_t(I^i) \text{ is disjoint from } Q = \{ q \} \)

so:

we didn’t quite prove yet but \( f \) is homotopic away maps to \( (X, B) \)
to a map to \((A, C)\), but we did almost as well: we proved it’s
homotopic away maps to \( (X, X-P) \) to a map to \((X-Q, X-(P \cup Q))\)

These are homotopy equivalent (collapsing \( \Delta_{\alpha}^{m+n} \{ p_{\alpha} \} \) to \( \{ q \} \))

so in the commutative diagram of inclusion maps

\[
\begin{array}{ccc}
\pi_1(A, C) & \xrightarrow{i_*} & \pi_1(X, B) \cup \{ q \} \\
\downarrow & & \downarrow \\
\pi_1(X-Q, X-Q-P) & \to & \pi_1(X, X-P)
\end{array}
\]

In last right group, \( [f] = [f_t] \text{ where } f_t \text{ came from lower-left} \)

\( \Rightarrow [f] \in \text{ im}(i_*) \).

(6) For injectivity: assume \( f_0, f_1 : (I^i, \partial I^i, J_i) \to (A, C, x_0) \) \( i < m+n \) represent same
element in \( \pi_1(X, B) \): then \( \exists \) homotopy \( F : (I^i, \partial I^i, J_i) \times [0,1] \to (X, B, x_0) \).

Deform \( F \) using PL approximation lemma as before; as above,
can find \( q \in \Delta_{\beta}^{i} \), \( p_{\alpha} \in \Delta_{\alpha}^{m+n} \), and a function \( \varphi : I^{i-1} \times [0,1] \to [0,1], \)
\( \varphi(I^{i-1} \times \{ 0 \}) = 0 \), whose graph separate \( F^{-1}(q) \) from \( \cup_{\alpha} F^{-1}(p_{\alpha}) \).
(Note: The dimension condition is now $i+1 \leq m+n$, i.e. $i < m+n$).

As before this allows us to excise $F^i(Q)$ from the domain of $F$,

define $F$ to a homotopy between $f_0, f_1$ among maps

$$( I^i, J^i, J_0 ) \to ( X, Q, X - ( P \cup B ) , x_0 )$$

(whence retracted onto $(A, C, x_0)$).

Hence $f_0, f_1$ reprent the same element of $\pi_i(A, C, x_0)$.

**Case 2:**

$A = C \cup (m+1)_\ast$ cells $e_{m+1}^i$ as in case 1

$B = C \cup$ cells of dim $\geq n+1$.

**Subj.:** Any $f: (I^i, J^i, J_0 ) \to ( X , B , x_0 )$ hits only finitely many cells (by compactness),

and using case 1 repeatedly we can pull it off the cells of $B - C$ one at a time (starting with highest-dim. cells).

**Inj.:** Similarly for $F: (I^i, J^i, J_0 ) \to ( X , B , x_0 )$ ....

**Case 3:**

$A = C \cup$ cells of dim $\geq m+1$

$B = C \cup$ cells of dim $\geq n+1$ as in case 2.

By cellular approx., can ignore cells of dim $> m+n+1$ in $A$ (don't affect $\pi_i$, is $m+n$)

let $A_k = C \cup$ cells of dim $\leq k$, $X_k = A_k \cup B$

Pure result for $\pi_i(A_k, C) \to \pi_i(X_k, B)$ by induction on $k$ starting at $k = m+1 \in$ case 2 and ending at $k = m+n+1$.

Look at (e.g. in rel. homotopy for triple $\langle A_k, A_{k-1}, C \rangle$ and $\langle X_k, X_{k-1}, B \rangle$): $\pi_{i+1}(A_k, A_{k-1}) \to \pi_i(A_{k-1}, C) \to \pi_i(A_k, C) \to \pi_i(A_k, A_{k-1}) \to \pi_i(A_{k-1}, C)$ $\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$\pi_i(X_k, X_{k-1}) \to \pi_i(X_{k-1}, B) \to \pi_i(X_k, B) \to \pi_i(X_k, X_{k-1}) \to \pi_i(X_{k-1}, B)$

for $i < m+n$, iso by case 2 iso by induction iso by case 2 iso by induction

$\Rightarrow$ by five lemma, middle map is iso. Conclude by induction

(for $i = m+n$, get surjection by one half of five lemma)

(for $i = 1$, argue directly instead).

**General case:** use CW-approximation to replace $(A, C)$ and $(B, C)$ by homotopy equivalent CW-pairs so all cells have dim $\geq m+1 > n+1$, i.e. reduce to case 3.

(Since here $(A, C) \simeq (A', C)$ and $(B, C) \simeq (B', C)$ are idem $C$, fit together to $A' \cup B' \simeq A \cup B$).
Example: we've seen above that \( \pi_r(S^n) \cong \mathbb{Z} \). In fact this lets us calculate
\
\[ \pi_n \left( \bigvee_{\alpha} S^n_{\alpha} \right) \cong \bigoplus_{\alpha} \pi_n(S^n_{\alpha}) = \bigoplus_{\alpha} \mathbb{Z} \quad (\text{for } n \geq 2) \]

Indeed, for a finite collection, \( \prod_{\alpha} S^n_{\alpha} = \left( \bigvee_{\alpha} S^n_{\alpha} \right) \vee \) (cells of dim \( \geq 2n \))
so \( \pi_n \left( \bigvee_{\alpha} S^n_{\alpha} \right) \cong \pi_n \left( \prod_{\alpha} S^n_{\alpha} \right) = \prod_{\alpha} \pi_n(S^n_{\alpha}) \vee \) only affects \( \pi_i \), \( i \geq 2n-1 \).

For infinite collection, recall any map \( S^n \to \bigvee_{\alpha} S^n_{\alpha} \) on any homotopy
only hits finitely many of the \( S^n_{\alpha} \), so get \( \bigoplus \pi_n(S^n_{\alpha}) \).

- For \( n \geq 2 \), \( \pi_n(S^1 \vee S^n) \) = free abelian gp w/ countably \( \infty \) generators
  indeed \( \pi_n(S^1 \vee S^n) \cong \pi_n(\text{univ. cover}) \), but univ. cover = \( \vee S^n \infty \)

Note \( \pi_r(S^1 \vee S^n) = \mathbb{Z} \) action is non-trivial.

Generator acts by \( \infty \times \) to \( \infty \times = \text{next generator} \), so in fact
\[ \pi_n(S^1 \vee S^n) \cong \mathbb{Z}[[t,t']] \text{ as a module over } \mathbb{Z}[t] \cong \mathbb{Z}[[t]]. \]

Another Corollary of excision:

Prop: If a C.W. pair \((X,A)\) is \( r \)-connected and \( A \) is \( s \)-connected, \( r,s \geq 0 \)
then the maps \( \pi_i(X,A) \to \pi_i(X/A) \) induced by quotient map \( X \to X/A \)
are isos: \( i \leq r+s \)
subjunct: \( i = r+s+1 \).

Pf: \( X \cup CA \) attach cone on \( A \) along \( A \)
\[ \begin{array}{c}
\text{CA contractible subcomplex} \\
\Rightarrow \quad X \cup CA \to (X \cup CA)/CA = X/A \quad \text{is a homotopy equivalence.} \\
\end{array} \]

Also: \( A \) \( s \)-connected \( \Rightarrow \) \((CA,A)\) \( s+1 \)-conned (since \( r.e.s = \pi_{i+1}(CA,A) \cong \pi_i(A) \))
Excision \( \Rightarrow \) inclusion induces \( \pi_i(X,A) \to \pi_i(X/(X \cup CA,CA)) \) isom. for \( i \leq r+s \)
\( \text{eqn. for } i = r+s+1 \)
Example: Eilenberg-Maclane space

Def. \( a \text{ k}(G, n) \) is a CW-plex \( K \) s.t.
\[
\pi_n(K) \cong G, \\
\pi_i(K) = 0 \text{ for } i \neq n.
\]

Construction: for \( n \geq 2 \) and \( G \) any abelian group:

1. First build an \((n-1)\)-complex CW-plex \( X \) with \( \pi_n(X) \cong G \):
   - Start w/ a presentation of \( G \) by generators \& relations, i.e. \( G = (\oplus \mathbb{Z})/H \).
   - Consider a wedge \( \bigvee_{\alpha} S^n_{\alpha} \) of spheres, one sphere for each generator \( \alpha \) \( \Rightarrow \pi_n = (\oplus \mathbb{Z}) \).
   - Let \( \varphi_\beta : S^n \to \bigvee_{\alpha} S^n_{\alpha} \) mapping generators of \( H = \text{Ker}(\oplus \mathbb{Z} \to G) \) (i.e. relations).
   - Attach \((n+1)\)-cells \( e^{n+1}_\beta \) along \( \varphi_\beta \) to get a CW-plex \( X \).
   - \( X \) is \((n-1)\)-conn. since only \((n)\)-cells \& \((n+1)\)-cells.

2. k.s. of pair \((X, \bigvee_{\alpha} S^n_{\alpha})\):
   - \( \pi_{n+1}(X, \bigvee_{\alpha} S^n_{\alpha}) \to \pi_n(\bigvee_{\alpha} S^n_{\alpha}) \to \pi_n(X) \to \pi_n(X, \bigvee_{\alpha} S^n_{\alpha}) \)
   - By last proposition:
   - \( \pi_{n+1}(X/\bigvee_{\alpha} S^n_{\alpha}) = \pi_{n+1}(\bigvee_{\beta} S^{n+1}) = (\oplus \mathbb{Z})_{\beta} \) (cofiber approx)
   - \( \pi_n(X) = (\oplus \mathbb{Z})/H = G \).

3. Next kill \( \pi_{n+1}(X) \) by attaching \((n+2)\)-cells along its generators w/o modifying \( \pi_n \).

4. Then kill \( \pi_{n+2}(X) \) by attaching \((n+3)\)-cells and so on... to get a \( \text{k}(G, n) \).

(For \( n = 1 \) & \( G \) any group, not nec. abelian:

Similarly, build a 2-dim. CWplex with \( \pi_1 \cong G \) by taking a wedge of \( S^1 \)'s for generators of \( G \) and attaching 2-cells along relations
then kill \( \pi_2, \pi_3, \ldots \) by attaching higher \( \dim \) cells.)
Ex: \[ S^n \text{ is } K(Z,1), \quad T^n = (S^1)^n \text{ is } K(Z^n,1) \] \[ \text{rip} \infty \text{ is } K(Z/Z,1) \]
\[ \text{rip} \infty \text{ is } K(Z,2) \text{ (w/n) e}. \]

Ex: can build \( X \) with \( \pi_n(X) = G_n \) arbitrary; take \( X = \prod_n K(G_n,n) \).

Prop: Any two \( K(G,n) \) complexes are homotopy equivalent.

Pf: we'd like to use Whitehead's thm. but need to make sure the iso of \( \pi_n \) is.

induced by some actual map. Useful lemma:

Lem: \[ X = \left( \bigcup S^n_x \right) \cup Y \quad (\text{as in above construction}, \; n \geq 1) \]
\[ \text{Then } \forall \text{ homomorphisms } \psi: \pi_n(x) \to \pi_n(y), \; \exists f: X \to Y \text{ s.t. } f \circ \pi = \psi. \]

Pf: Recall \( \pi_n(X) = \text{(free gp gen by } S^n_x) / \text{(subgp gen by } \varphi_{\beta} : S^n \to S^n_x) \text{.)} \)

- map base point \( \to \) base point.
- over \( S^n_x \), let \( f \) be a map sending \( \psi([x]) \in \pi_n(y) \)

\( \tau_{\psi([x])} \in \pi_n(x) \) up by ind. of \( S^n_x \).

- to extend \( f \) over cell \( e_{\beta}^{n+1} \) w/ attaching map \( \varphi_{\beta} \), need to know that:

\[ f \circ \varphi_{\beta} : S^n \to S^n \] is nullhomotopic - true since \( f_{\ast}([\varphi_{\beta}]) = \psi([\varphi_{\beta}]) = \psi(0) = 0 \).

Hence extend to \( f: X \to Y \)

- by contr. \( f_{\ast}([x]) = \psi([x]) \) \( \forall x \), \( [x] \) generic \( \pi_n(X) \), so \( f \) = \psi.

So, consider the construction above building a \( K(G,n) \) \( K \) by attaching higher-dim.

handle to \( X = \left( \bigcup S^n_x \right) \cup \left( \bigcup e_{\beta}^{n+1} \right) \) to kill its \( \pi_n, \pi_{n+2}, \ldots \)

and let \( K' \) any other \( K(G,n) \).

By lemma, \( \exists \text{ map } f: X \to K' \) realizing isomorph \( \pi_n(X) = \pi_n(K') = G \).

To extend \( f \) to \( K \): for each \( (n+2) \)-cell \( e^{n+2} \) w/ attaching map \( \varphi: S^{n+1} \to X \)

\( f \circ \varphi \) is nullhomotopic in \( K' \) since \( \pi_{n+1}(K') = 0 \), hence \( f \) extends every cell.

Continue over all higher dim! Cells of \( K \), and obtain \( f: K \to K' \).

By Whitehead's thm, \( f \) is a homotopy eqv.