Whitehead's theorem: homotopy groups of CW-complexes carry a lot of information!

Thm.: If $f: X \to Y$ map be connected CW-complexes, $f_*: \pi_n(X) \to \pi_n(Y)$ isom. $\forall n$ \implies $f$ is a homotopy equivalence

(i.e. $\exists g: Y \to X$ s.t. $f \circ g \simeq id$ and $g \circ f \simeq id$).

If $f$ is inclusion of a subcomplex then stronger statement holds:

$X$ is a deformation retract of $Y$.

**Remark:** Whitehead's thm does not say that two spaces with isomorphic $\pi_n$ are homotopy equivalent! The isom. have to be induced by a map $f$.

**Ex:** $X = \mathbb{RP}^2 \times S^\infty (\simeq \mathbb{RP}^2)$ \hspace{1cm} $Y = S^2 \times \mathbb{RP}^\infty$

$\pi_1(X) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$ \hspace{1cm} $\pi_1(Y) = \pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$

$X = Y = S^2 \times S^\infty$ so same $\pi_n \forall n \geq 2$

However $H_k(X) = H_k(\mathbb{RP}^2) \neq 0$ only for $k \leq 2$ (recall $S^\infty$ contractible)

$H_k(Y) \neq 0$ for only many $k$ (since $\mathbb{RP}^\infty$ has only many non zero homology groups).

so $X \neq Y$!

**Remark:** similar statement for $H_k$ is false: take $X$ with $H_1(X) \cong M_p$ but $\pi_1(X) \neq 0$ (eg. Poincaré sphere minus a point), $f: X \to pt$ cont.

However there's a version of Whitehead's thm in homology assuming $X,Y$ simply connected (see below).

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**Proof:**

**Key technical lemma:** Compression lemma:

For $n=0$, $\pi_0 B \to \pi_0 X$\(\phi\)

$(X,A) \text{ CW pair, } (Y,B) \text{ any pair with } B \not\subset A$.

$\forall n \geq 1$ s.t. $X-A$ has $n$-dim. cells, assume $\pi_n(Y,B,y_0) = 0 \forall y_0 \in B$.

Then every map $f: (X,A) \to (Y,B)$ is homotopic rel $A$ to a map $X \to B$.

**Proof:** by induction on $n$. Assume $f$ has been homotoped so that $f$ maps the $(k-1)$-skeleton $X^{k-1}$ to $B$. Consider a $k$-cell $e^k$ of $X-A$, with characteristic map $\phi: D^k \to X$, then to $\phi: (D^k, B \cdot D^k) \to (Y,B)$.}

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**Lecture 4, Mon 1/30**
Since \( \pi_k(y, B, y_0) = 0 \), can hom-htpy for rel. \( x^k \) so it maps into \( B \) 
(by comparison criterion)

\( \Rightarrow \) induce homotopy of \( f \) on \( x^{k-1} \cup e^k \), rel. \( x^{k-1} \).

Doing this on all \( k \)-cells, get a homotopy of \( f|_{x^k \cup A} \) to a map into \( B \) 
(& not moving \( A \)).

By homotopy extension property for CW-pairs, can extend this homotopy to all \( x \).

Proceed by induction. (If \( \dim X \) do steps in time \( \frac{1}{2^k} \) so it converges

to a homotopy \( (0,1) \times X \to Y \), well defined since

any cell eventually stationary).

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**Proof of Whitehead's theorem:**

1) special case where \( f: X \to Y \) inclusion of subcomplex: consider long exact seq.

in relative homotopy:

\[
\cdots \to \pi_n(x) \xrightarrow{f_*} \pi_n(y) \to \pi_n(y, x) \xrightarrow{r_*} \pi_{n-1}(x) \xrightarrow{f_*} \pi_{n-1}(y) \to \cdots
\]

so \( \pi_n(y, x) = 0 \) \( \forall n \).

Then by comparison lemma, the identity map \( \text{id}: (y, x) \to (y, x) \)

is homotopic rel. \( x \) to a map \( r: Y \to X \), \( r|_X = \text{id}_X \)

ie. a deformation retraction of \( Y \) onto \( X \) (in particular a homotopy equiv.).

2) general case: consider mapping cylinder of \( f \).

\[
M_f = (X \times I) \cup Y
\]

\[
\text{\begin{tikzpicture}
\draw (0,0) -- (3,0) -- (3,2) -- (0,2) -- cycle;
\draw (0,0) -- (0,2);
\draw (3,0) -- (3,2);
\node at (1.5,1) {f(x)};
\end{tikzpicture}}
\]

\( M_f \) contains both \( X \times \{0\} = X \) and \( Y \) as subspaces

and retracts onto \( Y \) (see diagram).

\( f \) = composition of inclusion \( X \subset M_f \) and retraction \( \Pi_f \to Y \)

\( \Rightarrow \) enough to show \( X \subset M_f \) is a homotopy equiv.

Note: since \( f_* \) is iso for all \( \pi_n \) so is \( i_* \).

If \( f \) is a cellular map taking n-skeleton \( X^n \to Y^n \) \( \forall n \),

then \( M_f \) is clearly a cell complex and \( (M_f, X) \) is a CW-pair

so that follows from above special case.

Otherwise: either use a trick (see Hatcher) or cellular approximation thm:

\( f \) is homotopic to a cellular map. (will see soon).
Cellular approximation:

We'd like to prove $\pi_n(S^k) = 0$ for $n < k$ by just saying that

a map $S^n \to S^k$ must miss some point $q \in S^k$, then contract $S^k \setminus \{q\}$ to $\star$ ... but first need to ensure the map isn't surjective! (Space-filling curve...) 

In fact, when studying maps like CW-complexes, can reduce to

**cellular maps:** $f: X \to Y$ s.t. $f(X^n) \subseteq Y^n \forall n$ (maps n-cells to cells of dim $\leq n$).

Then (cellular approximation):

Every map $f: X \to Y$ of CW-complexes is homotopic to a cellular map.

If $f$ already cellular on subcomplex $A \subseteq X$, can take homotopy stabilizing on $A$.

Gally: $\pi_n(S^k) = 0$ for $n < k$.

(*pf:*) $S^k = 0$-cell $\cup$ cell $\quad$ Any map $S^n \to S^k$ can be homotoped to $S^n = 0$ cell $\cup$ no cell $\quad$ a cellular map, i.e. (constant map).

This is similar to simplicial approximation lemma for simplicial complexes (simplicial maps are cellular!) but doesn't require subdivision of the domain.

**pf of lemma:** By induction on dim. Assume $f$ cellular on $(n-1)$-skeleton $X^{n-1}$, let $e^n$ be an n-cell of $X$. Since $\bar{e^n} \subseteq X$ compact, $f(\bar{e^n})$ compact hence intersects finitely many cells of $Y$. (Snick topology: seq. of pts in $\bar{e^n}$ diff cell cannot converge in $Y$.)

Let $e^k \in \gamma$ cell of highest dim. meeting $f(e^n)$; if $k < n$, $f$ len already cellular $\forall k < n$.

So assume $k > n$; we'll show:

**Claim:** can homotopy $f|_{\partial^n} : e^n \to \gamma$ rel boundary so that it misses some point $p \in e^n$ only in $e^k$.

Then get $f(e^n)$ to miss all of $e^k$ by composing with a retraction of $Y \setminus p$ to $Y \setminus e^k$. Repeat (finitely many times) to eventually get $f(e^n)$ to miss all cells of dim $> n$.

Do this for all n-cells to get homotopy of $f|_{X^n} : \text{rel} \ X^{n-1} \cup A$ to cellular map.

By homotopy extension property for CW-pairs, can extend this homotopy to all $X$.

Prove by induction. (if dim $\infty$, do steps in cone $\frac{1}{2^n}$ so it converges to a homotopy $[0,1] \times X \to Y$, well def. since any cell eventually stabilizes.)
PF Claim: follows from a linear approximation lemma.

* Definition: polyhedron in $\mathbb{R}^n := \text{union of finitely many convex polyhedra} = \text{compact set, } \cap \text{ finitely many half spaces delimited by hyperplanes}

* PL (piecewise linear) map polyhedron $\to \mathbb{R}^k := \exists$ decomposed into convex polyhedra stk: linear on each

**Lemma:** $f : I^n \to \mathbb{R}^k = W v e^k$ of cell, then $f$ is homotopic rel $f^{-1}(W)$ to a map $f_1$ for which $f_1(k) \in \mathbb{R}^k$ is PL wrt $e^k = \mathbb{R}^k$

1. $f_1(k) \in e^k$, $f_1(k)$ is PL wrt $e^k = \mathbb{R}^k$
2. $K \supseteq f_1^{-1}(U)$ for some open $U \ni 0 \subset e^k$.

Apply this to our situation: for $k > n$, we have $f_1 : e^n = W v e^k, k > n$. Lemma gives map $f_1$ which only differs from $f$ inside $e^k$ and is PL on some $K \subset e^n$; since $f_1 | K$ can't be onto $U$ ($h, n$)

$\text{pf: Lemma.}$ Identify $e^k = \mathbb{R}^k, \mathbb{B}_1 \subset B_2 \subset e^k$ balls of radius 1 & 2.

$f^{-1}(B_2)$ closed $\subset I^n$, hence compact $\Rightarrow f$ is uniformly continuous on $f^{-1}(B_2)$

There exist $\epsilon \gt 0$ s.t. $d(f(x) - f(y)) < \epsilon/2 \Rightarrow |x - y| < \epsilon$.

Also can choose $\epsilon < \frac{1}{2}$ s.t. $\frac{1}{2}d(f^{-1}(B_1), I^n - f^{-1}(\text{Int} B_2))$ disjoint convex

Subdivide $I^n$ into cubes of diameter $< \epsilon$, $K_1 = U \text{ all cubes needed } f^{-1}(B_2)$, $K_2 = \text{all adjacent cubes to these}$.

Next, subdivide $K_2$'s cube into simplices ($\times \times \times$, implicitly)

$g : K_2 \to e^k$ a PL map that $= f$ at vertices of $K_2$ & linear on simplices

$Q : K_2 \to [0,1]$ with 1 at 0, $Q|_{K_1} = 1, Q|_{K_2} = 0$. Then set

$f_1 = (1-tQ) f + tQ g$ on $K_2$, $f$ elsewhere.

$f_1$ is $f$ outside of $K_2$ hence on $f^{-1}(W)$; $g$ is PL on $K_1$; since $|f_1 - f| < \epsilon/2$ so $f^{-1}(B_{1/2}) \subset K_1$.
Remark: can also use cellular approx for maps of pairs!

Every map \( f : (X,A) \rightarrow (Y,B) \) can be deformed through maps of pairs to a cellular map.

Indeed, first deform \( f|_A : A \rightarrow B \) to be cellular; then extend this to a homotopy of \( f \) on all of \( X \); then deform resulting map to a cellular one, remaining fixed on \( A \).

Consequence: Given a CW-pair \( (X,A) \),

If all cells of \( X-A \) have dim. \( \geq n \) then \( (X,A) \) is \( n \)-connected.

In particular \( (X,X^n) \) is \( n \)-connected. Hence \( \pi_k \rightarrow \pi_k \) induces isomorphisms on \( \pi_k \) \( k \leq n \), and surjection on \( \pi_n \).

Pf: Given a map \( (D^k, \partial D^k) \rightarrow (X,A) \), where \( k \leq n \), cellular approx defines it to a cellular map, which sends \( D^k \) to \( A \).

This proves \( \pi_n (X,A) = 0 \) (every map \( (D^k, \partial D^k) \rightarrow (X,A) \) defines into \( A \)).

The second statement follows from long exact seq. for relative \( \pi_k \).

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**CW-approximation:**

**Def:** \( f : X \rightarrow Y \) is a weak homotopy eqv if induces isoms. \( f_* \) on \( \pi_n (X,A) \)

\( \forall n \) and \( A \) base point.

By which, a weak homotopy eqv. between CW-complexes is a homotopy eqv.

Hence also true for spaces which are homotopy eqv. to CW-complexes.

However, \( \mathbb{E} \) spaces which have all homotopy groups trivial, but aren't contractible, eg "quasicycle" \( \Rightarrow \) there are weakly homotopy eqv. to a point, but not homotopy eqv. to it (as to any CW-complex).

So: in general, weakly h.e. \( \Rightarrow \) h.e.

**Def:** A CW-approximation of \( X \) is a CW-complex \( Z + \) a weak h.e. \( f : Z \rightarrow X \).

[Prop: A weak h.e. induces isom. on all homotopy, homology & cohom. groups; will follow from Hurewicz] so for many arguments, we can reduce to the case of CW-complexes!]

May want variants on this theme, eg. - CW-pair approx. \( (Z,A) \) of pair \( (X,A) \)

\( A \subset X \subset \text{cw-complex} \)

- approx. up to dim. / in dim. \( \geq n \) only.
Construction of CW-approx. of \((X,A)\): 

\[ \text{ACX}, \quad A \text{ CW-complex, } \pi_0(A) \to \pi_0(X) \] 

By induction, build \(Z\) a union of subcomplexes 

\[ A = Z_0 \subset Z_1 \subset \ldots \]

where \(Z_k\) obtained from \(Z_{k-1}\) by attaching \(k\)-cells.

- \(f: Z_k \to X\) identity on \(A\) and \(f_c\) isom. on \(\pi_i\), \(0 \leq i < k\) 
- \(f_c\) surj. on \(\pi_k\)

Given \(Z_k\), choose cellular maps \(y_c: S^k \to Z_k\) generating generators for the kernel of \(f_k: \pi_k(Z_k) \to \pi_k(X)\) (for all components fixing a basept in each).

Attach cells \(e^{k+1}_c\) to \(Z_k\) via \(y_c\), call resulting CW-complex \(Y_{k+1}\).

Since \(f_k\) surjective, can extend \(f\) over \(e^{k+1}_c\) to get \(f: Y_{k+1} \to X\).

**Claim:** \(f_k: \pi_k(Y_{k+1}) \to \pi_k(X)\) isom.

**Pf.**

- It is injective since each of kernel can be resolved by cellular maps, hence maps into \(Z_k\), and there are nullhomotopic in \(Y_{k+1}\), by construction.
- It still is surjective, since the surjective map \(\pi_k(Z_k) \to \pi_k(X)\) factors through a \(\pi_k(Y_{k+1})\).

and \(\pi_i, i < k\) are not affected by attaching \((k+1)\)-cells (cellular approx. is representatives of \(\pi_i\) and homotopies live in \((i+1)\)-skeleton).

- For \(k = 0\): \(Y_1\) = attach \(1\)-cells joining all basepts 0-cells of \(Z_0 = A\) which lie in same connected component of \(X\). Then clearly \(\pi_0(Y_1) \cong \pi_0(X)\).

- Next, choose \(y_\beta: S^{k+1} \to X\) generators of \(\pi_{k+1}(X)\) for given base pts in each component of \(X\) 

\[ Z_{k+1} = Y_k V(S^{k+1})_\beta \] 

(wedge sphere \(S_{k+1}\) at base pt of appropriate component)

and extend \(f\) by \(y_\beta\) on \(S^{k+1}\).

This guarantees surjectivity of \(f_k\) on \(\pi_{k+1}(Z_{k+1}) \to \pi_{k+1}(X)\).

Moreover, for \(i < k\), we have that \(\pi_i(Y_{k+1}) \cong \pi_i(Z_{k+1})\) \(\cong \pi_i(Z_k)\) and \(\pi_i(Y_{k+1}) \cong \pi_i(Z_k)\) because composition to \(\pi_i(X)\) is injective.

**Taking \(Z = \bigcup Z_k\):** Since maps \(\pi_i(Z) \to \pi_i(X)\) only depend on \((i+1)\)-skeleton, they are isos as well.
When \((X, A)\) is \(n\)-connected, by exact seq \(\pi_k(A) \cong \pi_k(X)\) \(k \leq n\),
so can skip first \(n\) steps of construction...

Contrary: \((X, A)\) \(n\)-connected \(\Rightarrow\) \(\exists\) CW-complex \((\mathbb{Z}, A)\) s.t. all cells of \(\mathbb{Z} - A\) have dimension \(\geq n+1\).

Can also choose to start at step \(n\) no matter what, to build a CW-complex that looks "Like A up to \(\pi_n\) and Like X after \(\pi_n\)." (see Hatcher Prop 4.13)

Example of similar hack: Poincaré towers

| Step | \(X\) CW complex (connected \(\forall n\)) \(\Rightarrow\) \(\exists\) CW-complex \(X_n\) with \(\pi_i(X_n) = \begin{cases} \pi_i(X) & i \leq n \\ 0 & i > n \end{cases}\)
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<td>and then fit into a tower + comm. diagram</td>
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Namely: to build \(X_n\), start from \(X\) and take \(q_\ast : S^{n+1} \to X\) yielding \(\pi_{n+1}(X)\),

Attach \((n+2)\)-cells \(e^n\) along \(q_\ast\) to get a CW-complex \(Y\).

Cellular approx. \(\Rightarrow\) \(\pi_i(X) \cong \pi_i(Y)\) for \(i \leq n\)

But \(\pi_{n+1}(Y) = 0\): by cellular approx. any \(S^{n+1} \to Y\) can be homotyped into \(X\), then by combination it's nullhomotopic in \(Y\).

Then attach \((n+3)\)-cells to \(Y\) along \(\operatorname{ker} \pi_{n+2}(Y)\) to kill it, and so on \(\ldots\) to get \(X_n\).

The inclusion \(X \subset X_n\) extends to a map \(X_{n+1} \to X_n\) since \(X_{n+1}\) obtained by attaching cells of \(dn \geq n+3\) to \(X\), and

\(\forall k \geq n+3\), \(\pi_{k+1}(X_n) = 0 \Rightarrow\) attaching map of \(k\)-cell is nullhomotopic

\(n\) \(\Rightarrow\) \(i\) \(\Rightarrow\) \(X_{n+i}\).