Homotopy groups: \( \pi_n(X) \) generalizes \( \pi_1 \): homotopy classes of based loops \( (S^1, x_0) \rightarrow (X, x_0) \) to homotopy classes of based maps \( (S^n, x_0) \rightarrow (X, x_0) \).

At first glance, may seem similar to \( H_n \), and indeed closely related, but still quite different… e.g. \( H_2(S^n) \) easy to compute, \( \pi_2(S^n) \) very hard (and still open problem!!)

**Definition.** Let \( I^n = [0,1]^n \) unit cube, \( X \) space with base point \( x_0 \),

\[
\pi_n(X, x_0) = \text{homotopy class of maps } f: (I^n, \partial I^n) \rightarrow (X, x_0).
\]

(Where homotopies should satisfy \( f_t(\partial I^n) = x_0 \forall t \))

Remarks:
- For \( n = 1 \), agree with \( \pi_1 \). (Intervals w/ both ends at \( x_0 \) \( \Rightarrow \) loops)
- For \( n = 0 \), by convention \( I^0 = \text{point} \), \( \partial I^0 = \emptyset \),

\[
\pi_0(X, x_0) = \{ \text{path components of } X \}
\]

For \( n \geq 2 \), define a sum operation on \( \pi_n \) (extend product on \( \pi_1 \)):

\[
\begin{align*}
\text{Def.:} & \quad (f+g)(s_1, \ldots, s_n) = \left\{ \begin{array}{ll} 
  f(2s_1, s_2 \ldots s_n) & s_1 < \frac{1}{2} \\
  g(2s_1 - 1, s_2 \ldots s_n) & s_1 \geq \frac{1}{2}
\end{array} \right. \\
= & \quad (-f)(s_1, \ldots, s_n) = f(1-s_1, s_2 \ldots s_n)
\end{align*}
\]

Agree with def. on \( \pi_1 \).
define a group (identity = constant map \( \text{in} \to x_0 \))
(associativity up to homotopy: same as \( f \circ (g \circ h) \sim f \circ (g \circ h) \))
also \( f + (-f) \sim \text{id} \)

However: \( f \) for \( n \geq 2 \), + is commutative & \( \pi_n(X, x_0) \) is an abelian group

\[
\begin{array}{c}
\text{if } f, \ g \\
\begin{array}{c}
\begin{array}{c}
\text{f + g} \\
\end{array} \\
\end{array} \\
\end{array}
\end{array}
\]

shrink domains of \( f \& g \)
to smaller cubes inside \( \text{in} \)
(map \( \equiv x_0 \) outside)

delete subsets around each other

**Two other useful viewpoints on \( \pi_n \):**

1) \( \text{in} / \partial \text{in} \approx \text{in} / \partial \text{in} \approx S^n \), so maps \( (S^n, \partial S^n) \to (X, x_0) \)

are the same as maps \( (\text{in}, \partial \text{in}) \to (X, x_0) \)

hence \( (S^n, x_0) \to (X, x_0) \).

- addition is then: \( S^n \xrightarrow{c} S^n \cup S^n \xrightarrow{f \lor g} X \)

2) Can think of \( \pi_2 \) as loops of based loops in \( X \) \( (\gamma_k(s) = f(t, s), x_0) \)

\( S_0 \xrightarrow{\text{reflect equator}} S_0 \\xrightarrow{f \lor g} X \)

loop space \( \Omega X = \text{based loops in } (X, x_0) = \{ \text{maps } (I, \partial I) \to (X, x_0) \} \)

w/ base pt = const loop compact-open topology \( U_{k, W} = \{ f | f(k) \in W \} \) (local uniform topology)

By def. of sum operation, this is a group isomorphism.

Similarly, \( \pi_n(X) \cong \pi_n(\Omega X) \) \( (\gamma(t_1, ..., t_n))(s) = f(t_1, ..., t_{n-1}, s) \)

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**Dependence on basepoint:** assume \( X \) path connected; \( \delta : I \to X \) path \( x_0 \)

\[
\begin{array}{c}
\text{Induce an isomorphism } \delta_* : \pi_0(X, x_1) \cong \pi_0(X, x_0)
\end{array}
\]

with \( \delta_* \cdot \eta_* = (\delta \cdot \eta)_* \) and \( (\delta^* \eta)_* = (\eta^* \delta)_* \)
Proof: given $f: (X, x_0) \to (Y, y_0)$, define $\gamma \circ f = (f \text{ in subarc,}
\text{interpolate } x_0 \to x_1 \text{ radially along } \gamma).
\text{Clearly } (\gamma \circ f) = \gamma \circ (\gamma \circ f).
\text{Claim } (\gamma \circ f) + (\gamma \circ g) = \gamma \circ (f + g).
\text{So } [f] \mapsto [\gamma \circ f] \text{ (clearly well-def.) is a group homomorphism } \gamma_g.
\text{Clearly } \delta_1 \cdot \gamma_1 = (\gamma_1)_*; \delta_k \text{ is an isom because } (\gamma_1)_* \gamma_1 = \text{id}.

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So can write $\pi_n(X)$ & forget base pt ... but not too much! as in the case of $\pi_1$,
these iss are noncanonical! in fact, considering case where $g$ = loop in $(X, x_0)$, get
an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ by automorphisms!
(for $n = 1$, $\pi_1([f]) = \frac{[gfg^{-1}]}{[g]}$ is conjugation by $[g]$ in $\pi_1$).
This make $\pi_n(X)$ a $\mathbb{Z}[\pi_1(X)]$ module for $n \geq 2$.
(Even through $\pi_n$ is abelian, it can still have a nontrivial module structure over $\pi_1$.)
(Exercise: $\pi_2(S^2 \vee S^1)$? module structure?)

Basic properties of $\pi_n$:
- $\pi_n$ is a functor, i.e. a map $\varphi: (X, x_0) \to (Y, y_0)$ induces $\varphi_*: \pi_n(X, x_0) \to \pi_n(Y, y_0)$
  $\varphi_* [f] = [\varphi \circ f]$.
  Clearly, $\varphi_* \circ \gamma = \gamma_* \circ \varphi$; $(\varphi \circ \varphi)_* = \varphi_* \circ \varphi_*$. 
  In particular a homotopy equivalence induces isomorphism $\pi_n$.
• Covering spaces:

Prop: \( \forall n \geq 2, \pi_n(X, x_0) \rightarrow \pi_n(\tilde{X}, \tilde{x}_0) \) is an isomorphism

Pf: every map \( (S^n, x_0) \rightarrow (X, x_0) \) lifts to \( \tilde{X} \) for \( n \geq 2 \) (\( S^n \) simply connected) (uniquely). This yields \( \sim \) \( \forall n \geq 2 \).

In particular, if the universal cover \( \tilde{X} \) of \( X \) is contractible then \( \pi_n(X) = 0 \) \( \forall n \geq 2 \).

E.g.: for the torus \( T^n = (S^1)^n \), the universal cover is \( \mathbb{R}^n \), so \( \pi_i(T^n) = 0 \) \( \forall i \geq 2 \).

(say \( T^n \) aspherical).

• Products:

Prop: \( \pi_n(\prod_{i \in I} X_i) \cong \prod_{i \in I} \pi_n(X_i) \)

Pf: maps \( S^n \rightarrow \prod_{i \in I} X_i \) \( \leftrightarrow \) collection of maps \( f_i: S^n \rightarrow X_i \) \( \forall i \)

same for homotopy \( S^n \times I \rightarrow \prod_{i \in I} X_i \) \( \leftrightarrow \) collection of homotopies.

Again much simpler than homology (Künneth formula).

Relative homotopy groups: for a pair \( (X, A) \) and base point \( x_0 \in A \):

\[
\begin{align*}
I^n &= \{ x \in D^n : \text{dist}(x, \partial D^n) < 1 \} \\
I^{n-1} &= I^n \setminus \{ x \}
\end{align*}
\]

\[
\begin{align*}
\pi_0^n(X, A, x_0) &= \text{homotopy classes of maps} \\
(I^n, \partial I^n, J_n) ightarrow (X, A, x_0)
\end{align*}
\]

Alternatively, since \( (I^n, \partial I^n) / J_n \cong (D^n, S^{n-1}) \), can think of

\( \pi_0^n(X, A, x_0) \) as homotopy classes of maps \( (D^n, \partial D^n = S^{n-1}, x_0) \rightarrow (X, A, x_0) \)

(sbee \( \pi_2(X, A) \) is degree in \( X \) w/ boundary in \( A \)).

Addition: as before, concatenate in 1st coordinate.

Composition trick only works for \( n \geq 3 \) though (can't use \( n \)-th coordinate as before)

so:

\[
\pi_n(X, A, x_0) = \begin{cases} 
\text{set for } n = 1 \\
\text{group for } n = 2 \\
\text{abelian gp for } n \geq 3
\end{cases}
\]
Compression criterion: \( f: (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0) \) represents 0 in \( \pi_n(X, A, x_0) \)
iff it is homotopic rel \( S^{n-1} \) to a map with image \( \subseteq A \)

**PF:** if \( f \sim g \) with \( g(D^n) \subseteq A \) then \([f] = [g]\) and \([g] = 0\) since can compose \( g \) with deformation retraction of \( D^n \) to \( s_0 \) to get \( g \sim \text{cont map} \).

- Conversely, if \( f \sim 0 \) via homotopy \( F: D^n \times [0,1] \rightarrow X \)
  then compose \( F \) with homotopy \( \begin{array}{c}
  D^n \\
  \uparrow \downarrow
  \end{array} \xrightarrow{i \circ j} S^{n-1} \times \{0,1\} \cup D^n \times \{1\} \)
  to get a homotopy from \( f \) to map w/ values in \( A \) stationary on \( S^{n-1} \).

- \( \phi: (X, A, x_0) \rightarrow (Y, B, y_0) \) induces homomorphisms on \( \pi_n \), as before.

**Long exact sequence:** (cf. relative homology!)

**Thm:**

\[ \ldots \rightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \ldots \rightarrow \pi_0(X, x_0) \]

- \( i_*: A \subseteq X \) inclusion
- \( \partial: \pi_n(X, A, x_0) \rightarrow \pi_{n-1}(A, x_0) \) restriction to boundary (from \( i^{n-1} \) to \( i^{n-1} \) or \( D^n \) to \( S^{n-1} \)).

(\& similarly for a triple \((X, A, B)\) w/ \( x_0 \in B \subseteq A \subseteq X \), relative \( \pi_n \)'s).

**PF:**
- exactness at \( \pi_n(X, x_0) \): \( j_* \circ i_* = 0 \) by compression criterion (maps to \( A \) represent zeros in \( \pi_n(X, A) \)).

Assume \( f: (I^n, \partial I^n) \rightarrow (X, x_0) \) represents zero in \( \pi_n(X, A) \), then by compression criterion can homotope \( f \) (keeping \( \partial I^n \rightarrow x_0 \)) to a map \((I^n, \partial I^n) \rightarrow (A, x_0) \) hom \([f] \in \text{Im } i_*\).

- exactness at \( \pi_n(X, A) \): \( \exists \delta j_\delta = 0 \) since boundary reduction of a map \((I^n, \partial I^n) \rightarrow (X, x_0) \) is cont; conversely, assume \( f: (I^n, \partial I^n, j_n) \rightarrow (X, A, x_0) \) has \( \partial [f] = 0 \), i.e. \( f|_{I^{n-1}} \subseteq \text{cont} \), by homotopy \( F: I^{n-1} \times [0,1] \rightarrow A \), then \( f \) is homotopic rel \( \partial I^{n-1} \sim x_0 \) maps \((I^n, \partial I^n, J_n) \rightarrow (X, x_0) \) to \( \begin{array}{c}
  x_0 \\
  \downarrow f \\
  x_0
  \end{array} \xrightarrow{\sim} \begin{array}{c}
  \delta f: (J^n, \partial J^n) \rightarrow (X, x_0)
  \end{array} \)

(stuck \( f \) with homotopy of \( F \))
exactness at \( \pi_n(A, x_0) \): \( i_*i^* = 0 \) since neither \( i \) nor \( i^* \) of \( f : X \to A \) is homotopic rel. \( A \) to constant map at \( x_0 \) itself!

Conversely: if \( i^*_*(f_*) = 0 \) then \( f : (X^1, \partial X^1, J_{n+1}) \to (A^1, \partial A^1, J_n) \)

\[ f : (I^{n+1}, \partial I^{n+1}, J_{n+1}) \to (X, A, x_0) \]

is homotopic rel. \( \partial I^n \) to constant map in \( X \)

The homotopy give \( F : (I^{n+1}, \partial I^{n+1}, J_{n+1}) \to (X, A, x_0) \), \( i_*([F]) = [f_0] \).

Ex.: Cone \( C_X = X \times I / X \times \{0\} \)

\( C_X \) contractible \( \Rightarrow \) get \( \pi_n(C_X) \to \pi_n(CX, x) \to \pi_{n-1}(X) \to \pi_n(CX) \)

\( 0 \to 0 \) isom.

The 0.c.s. in natural with maps of pairs; and change of basepoints induce isomorphisms on relative \( \pi_n \)'s (\( x, f = \) )

**Def.** \((x, x_0) \) is \( n \)-connected if \( \pi_k(x, x_0) = 0 \) \( \forall k \leq n \).

- 0-connected = path conn. \( (\Rightarrow x_0 \) doesn't matter
- 1-connected = simply conn.

\( \pi_k(x, x_0) = 0 \) \( \forall x_0 \in X \) \( \Rightarrow \) every map \( S^k \to X \) is homotopic to a constant map

\( \Rightarrow \) every map \( S^k \to X \) extends to a map \( D^{k+1} \to X \)

Similarly for pairs,
\( \pi_k(x, A, x_0) = 0 \) \( \forall x_0 \in A \)

\( \Rightarrow \) every map \( (S^k, S^{k-1}) \to (x, A) \) is homotopic rel. \( S^{k-1} \) to a map \( D^k \to A \)

\( \Rightarrow \) every such maps to a map \( D^k \to A \)

Say \((x, A) \) is \( n \)-connected if \( \pi_k(x, A) = 0 \) \( \forall k \leq n \).