1

<u>Recall:</u> $X_\psi := \{ (x_0 \dots x_4) \in \mathbb{P}^4 / f_\psi = \sum_0^4 x_i^5 - 5\psi\, x_0 x_1 x_2 x_3 x_4 = 0 \}$

$G = \{ (a_0 \dots a_4) \in \mathbb{Z}/5^5 \mid \sum a_i = 0 \} / \{(a,a,a,a,a)\} \simeq (\mathbb{Z}/5)^3$

acts by diagonal mult. by ξ^{a_i}, $\xi = e^{2\pi i/5}$

$\check{X}_\psi$ = crepant resolution of X_ψ / G LCSL as $z = (5\psi)^{-5} \to 0$

Last time we defined a G-inv't holom. vol. form Ω_ψ on X_ψ ($\leadsto \check{\Omega}_\psi$ on $\check{X}_\psi$) & computed its period on a 3-torus $T_0 \subset X_\psi$ ($\check{T}_0 \subset \check{X}_\psi$)

We got $\displaystyle\int_{T_0} \Omega_\psi = -(2\pi i)^3 \sum_{n=0}^\infty \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}$

or in terms of $z = (5\psi)^{-5}$, proportional to $\displaystyle\phi_0(z) = \sum_{n=0}^\infty \frac{(5n)!}{(n!)^5} z^n$

and we observed ϕ_0 solves a 4^{th} order diff. eqn, the <u>Picard-Fuchs eqn</u>:

$$\Theta^4 \phi = 5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\phi \quad \text{where } \Theta = z\frac{d}{dz}$$

<u>Prop:</u> all periods $\int_C \check{\Omega}_\psi$ also satisfy this equation

Simple reason why all periods satisfy <u>some</u> diff. equation:

$H^3(\check{X}_\psi)$ is 4-dimensional, so $[\check{\Omega}_\psi], \left[\frac{\partial \check{\Omega}}{\partial \psi}\right], \dots, \left[\frac{\partial^4 \check{\Omega}}{\partial \psi^4}\right]$ must be linearly related

$\Rightarrow$ so are their $\int$ over any 3-cycle.

$\Rightarrow \int_C \check{\Omega}_\psi$ solves 4^{th} order diff. eq^n.

* How to prove it: express Ω_ψ & its derivatives as residues

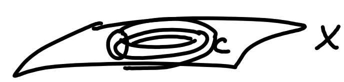
Let $\tilde{\Omega} = \sum_{i=0}^4 (-1)^i x_i \, dx_0 \wedge \dots \widehat{dx_i} \dots dx_4$

This is not a well-def'd 4-form on $\mathbb{P}^4$ because it's homogeneous of deg. 5 not 0 but if f, g homogeneous, $\deg f = \deg g + 5$, then $\frac{g\tilde{\Omega}}{f}$ is a global meromorphic 4-form on $\mathbb{P}^4$, with poles where $f = 0$.

② Ex: $\frac{5\psi\overline{\Omega}}{f_\psi}$ with poles along X_ψ.

Now, if we have a 4-form with poles along X, it has a <u>residue</u> on X – ideally a 3-form on X, or at least a class in $H^3(X, \mathbb{C})$.

$\mathrm{Res}_X\left(\frac{g\overline{\Omega}}{f}\right)$ s.t. $\forall$ 3-cycle C in X,



let Γ = "tube" 4-cycle = preimage of C in ∂(tub. nbd)

then $\frac{1}{2\pi i}\int_\Gamma \frac{g\overline{\Omega}}{f} = \int_C \mathrm{Res}_X\left(\frac{g\overline{\Omega}}{f}\right)$.

If we have simple poles along $X = f^{-1}(0)$, can get a 3-form: charact^d by

$$\mathrm{Res}_X\left(\frac{g\overline{\Omega}}{f}\right) \wedge df = g\overline{\Omega} \quad \text{at every point of } X$$

Then $\Omega_\psi = \mathrm{Res}_{X_\psi}\left(\frac{5\psi\overline{\Omega}}{f_\psi}\right)$ (compare w/ definition last time)

$\rightsquigarrow$ differentiating k times, $\frac{\partial^k}{\partial\psi^k}\Omega_\psi = \mathrm{Res}_{X_\psi}\left(\frac{g_k\overline{\Omega}}{f_\psi^{k+1}}\right)$

so ... compute $\Theta^4\Omega_\psi$ and $5z(5\Theta+1)\dots(5\Theta+4)\Omega_\psi$ where

$\Theta = z\frac{d}{dz} = \frac{1}{5}\psi\frac{d}{d\psi}$ in each form. Then show residues equal.

To compare residues of forms with order 5 poles along X_ψ, need algorithm for pole order reduction [Griffiths]:

Namely φ 3-form (w/ poles of order ℓ along X_ψ)

$$\varphi = \frac{1}{f_\psi^\ell}\sum_{i<j}(-1)^{i+j}(x_i g_j - x_j g_i)\, dx_0 \wedge \dots \widehat{dx_i}\, \widehat{dx_j} \dots dx_4$$

$g_0 \dots g_4$ degree $5\ell - 4$

$$\Rightarrow d\varphi = \frac{1}{f_\psi^{\ell+1}}\left(\ell\sum_j g_j\frac{\partial f_\psi}{\partial x_j} - f_\psi\sum_j\frac{\partial g_j}{\partial x_j}\right)\overline{\Omega}$$

so $\left(\sum g_j\frac{\partial f_\psi}{\partial x_j}\right)\frac{\overline{\Omega}}{f_\psi^{\ell+1}}$ can be rewritten as (lower order pole) + (exact)

doesn't affect residue.

criterion: top order term $\in$ Jacobian ideal gen^d by $\frac{\partial f_\psi}{\partial x_j}$'s. $\Rightarrow$ can reduce.

Apply pole order reduction to $\Theta^4\Omega_\psi - 5z(5\Theta+1)\dots(5\Theta+4)\Omega_\psi$, show $[\mathrm{Res}] = 0$.
(easier with computer algebra software).

③ Now we can find other periods of $\check{\Omega}_\psi$ using the theory of

<u>diff'l equation with regular singular pts</u> = diff. eqn. of the form

$$\Theta^s f + \sum_{j=0}^{s-1} B_j(z)\Theta^j f = 0 \qquad \text{where } \Theta = z\frac{d}{dz}$$

B_j holomorphic at $z=0$.

- Reduce to a 1st order system: let

$$A(z) = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & 0 & 1 \\ -B_0(z) & \cdots & & -B_{s-1}(z) \end{pmatrix}, \quad w(z) = \begin{pmatrix} f(z) \\ \Theta f(z) \\ \vdots \\ \Theta^{s-1} f(z) \end{pmatrix}$$

Then eqn becomes $\Theta w(z) = A(z)\, w(z)$.

<u>Thm:</u> $\exists$ constant $s\times s$ matrix R and $s\times s$ matrix of holom. functions $S(z)$

s.t.
$$\Phi(z) = S(z)\exp((\log z)R) = S(z)\left(Id + (\log z)R + \frac{(\log z)^2}{2}R^2 + \dots\right)$$

is a fundamental system of sol'ns for $\Theta w(z) = A(z)w(z)$.

Moreover, if $A(0)$ doesn't have eigenvalues differing by a nonzero integer then can take $R = A(0)$.

<u>NB:</u> Φ is <u>multivalued</u>! $z \mapsto e^{2\pi i}z$ gives $\Phi(z) \mapsto \Phi(z)e^{2\pi i R}$

so the monodromy is $e^{2\pi i R}$

In our case: $\mathcal{D}\phi = \Theta^4\phi - 5z(5\Theta+1)\dots(5\Theta+4)\phi = 0$

coeff of Θ^4 is $1-5^5z$

Coeff's of $\Theta^{i\le 3}$ are const. z

eqn rewrites as: $\Theta^4\phi - \dfrac{5z}{1-5^5z}P_3(\Theta)\phi = 0$

indep't of z

This is of the desired form, and $A(0) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ for $z=0$.

nilpotent $\Rightarrow$ assumption satisfied.

④

So the monodromy is given by $T = e^{2\pi i A(0)}$ unipotent of max. order

$$= \begin{pmatrix} 1 & 2\pi i & \frac{(2\pi i)^2}{2} & (2\pi i)^3/6 \\ & \ddots & \ddots & (2\pi i)^2/2 \\ & & \ddots & 2\pi i \\ & & & 1 \end{pmatrix}$$

In particular: 1^{st} column of Φ is inv't under $\Phi \mapsto \Phi T$: single valued sol.
the others are multivalued. (2^{nd} column $\mapsto$ itself $+ (2\pi i) \cdot (1^{st}$ col.)

<u>Relevance</u>: if $\omega(z) = \int_\beta \Omega$ is a period then it's a sol^n to Picard-Fuchs
$\Rightarrow$ it's a linear combination of fund.^{l} solutions
= first row of matrix $\Phi(z)$

so $\exists$ basis $\alpha_1 \ldots \alpha_4$ of $H_3(\check{X}, \mathbb{C})$ s.t. $\int_{\alpha_i} \Omega = \Phi(z)_{1i}$
The monodromy transformation in this basis is then
$T = \exp(2\pi i A(0))$ — this proves $z=0$ is max. unipotent (LCSL)

- <u>More periods of Ω</u>: we already have a sol^n $\phi_0(z)$ which is analytic, single-valued. By above, it's the only one up to scaling.

Next we'd like a multivalued solution $\phi_1(z)$ s.t.
$$\phi_1(z e^{2\pi i}) = \phi_1(z) + 2\pi i\, \phi_0(z)$$
($\leftrightarrow$ desired behavior for next fundamental sol^n! $\to$ unique up to $\phi_1 \sim \phi_1 + c\phi_0$.
& up to scaling, for period of Ω on β_1 s.t. $\beta_1 \xmapsto{\text{monodromy}} \beta_1 - m\beta_0$).

Necess: $\phi_1(z) = \phi_0(z) \log z + \tilde{\phi}(z)$, $\tilde{\phi}$ holomorphic

Let's find $\tilde{\phi}$. First note: $\Theta^i (f(z) \log z) = (\Theta^i f(z)) \log z + i\, \Theta^{i-1} f(z)$
(because $\Theta = z\frac{\partial}{\partial z} = \frac{\partial}{\partial \log z}$; product rule or induction)

so: if we write $F(x) = x^4 - 5z(5x+1)\cdots(5x+4)$, then
$$\mathcal{D}\phi_1(z) = F(\Theta)\big(\phi_0(z) \log z + \tilde{\phi}(z)\big)$$
$$= \underbrace{\big(\mathcal{D}\phi_0(z)\big)}_{=0} \log z + F'(\Theta)\, \phi_0(z) + \mathcal{D}\tilde{\phi}(z)$$

5

$\Rightarrow \underset{\text{5th order}}{\mathcal{D}\tilde{\phi}(z)} = \underset{\text{3rd order}}{-F'(\Theta)\,\phi_0(z)}$ gives a recurrence relation on the Taylor coefficients of $\tilde{\phi}$

calculate explicitly ... $\rightarrow$ $\tilde{\phi}(z) = 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{j=n+1}^{5n} \frac{1}{j} \right) z^n$

Now canonical coordinate: recall $\beta_0, \beta_1 \in H_3(\check{X}_\psi, \mathbb{Z})$, $\underset{\text{monodromy}}{\beta_1 \mapsto \beta_1 + \beta_0}$

Then $\int_{\beta_0} \check{\Omega} = C\,\phi_0(z)$

while $\int_{\beta_1} \check{\Omega} = C'\,\phi_0(z) + C''\phi_1(z)$

monodromy acts: $C'\phi_0 + C''\phi_1 \longmapsto C'\phi_0 + C''(\phi_1 + 2\pi i\,\phi_0)$

want $\int_{\beta_1} \check{\Omega} \longmapsto \int_{\beta_1+\beta_0} \check{\Omega}$

$\Rightarrow 2\pi i\, C'' = C.$

Then canon. coords: $w = \dfrac{\int_{\beta_1} \check{\Omega}}{\int_{\beta_0} \check{\Omega}} = \dfrac{C'}{C} + \dfrac{1}{2\pi i} \dfrac{\phi_1}{\phi_0}$

$= \frac{1}{2\pi i} \log c_2 + \frac{1}{2\pi i} \log z + \frac{1}{2\pi i} \frac{\tilde{\phi}(z)}{\phi_0(z)}$

$q = \exp(2\pi i w) = c_2\, z \exp\left(\frac{\tilde{\phi}(z)}{\phi_0(z)} \right)$

× constant because don't know β_1 for MS statement, only up to β_1 + mult. of β_0

can write power series ....