1 • Recall from last lecture: Large cx. structure limit degenerations

$\rightsquigarrow$ Canonical coordinates on complex moduli space

we have a basis $(\alpha_0 \ldots \alpha_s, \beta_0 \ldots \beta_s)$ of $H_3(X)$

β_0 invariant under monodromy, $\beta_1 \ldots \beta_s$ mapped by $\beta_i \overset{\gamma_j}{\longmapsto} \beta_i - \underset{\mathbb{Z}}{m_{ji}} \beta_0$

Normalize $\int_{\beta_0} \Omega = 1$: then $w_i := \int_{\beta_i} \Omega$, $\quad w_i \overset{\gamma_j}{\longmapsto} w_i - m_{ji}$

$q_i := \exp(2\pi i w_i)$ canonical coords.

Ex: for family of tori with monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\int_a \Omega = 1$, $\int_b \Omega = \tau$

$q = \exp(2\pi i \tau)$

• Earlier, we've seen: e_i basis of $H^2(\check{X}, \mathbb{Z})$, $e_i \in$ Kähler cone

$\rightsquigarrow$ coords. on complexified Kähler moduli space:

if $[B + i\omega] = \sum \check{t}_i e_i$, let $\check{q}_i = \exp(2\pi i \check{t}_i) \in \mathbb{C}^*$

(ie. $\check{t}_i = \int_{e_i^*} B + i\omega$)

Ex: for T^2, $\check{q} = \exp(2\pi i \cdot \int_{T^2} B + i\omega)$

Mirror symmetry statement:

$f: \mathcal{X} \to (D^*)^s$ family of CY 3-folds with LCSL point at 0.

Then $\exists$ CY 3-fold $\check{X}$ + $\exists$ choice of bases $\alpha_0 \ldots \alpha_s, \beta_0 \ldots \beta_s$ on $H_3(X)$

$e_1 \ldots e_s$ on $H^2(\check{X})$

s.t. under the map $m: (D^*)^s \longrightarrow \mathcal{M}_{\text{Käh}}(\check{X})$

$(q_1 \ldots q_s) \longmapsto (\check{q}_1 \ldots \check{q}_s)$ in canonical coordinates

the Yukawa couplings $\left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right\rangle_p = \left\langle \frac{\partial}{\partial \check{q}_1}, \frac{\partial}{\partial \check{q}_2}, \frac{\partial}{\partial \check{q}_3} \right\rangle_{m(p)}$

(2,1) Yukawa coupling: $\int_X \Omega \wedge \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} \Omega$

at point p given by periods q_i

(1,1) Yukawa coupling

ie. GW invts.

$2\pi i \, \check{q}_i \frac{\partial}{\partial \check{q}_i} = \frac{\partial}{\partial \check{t}_i} = e_i \in H^2(\check{X}, \mathbb{Z})$

② **Now:** computing canonical coords & (2,1)-coupling on mirror quintics.

Recall: $X_\psi := \left\{ (x_0 \dots x_4) \in \mathbb{P}^4 \,/\, f_\psi = \sum_0^4 x_i^5 - 5\psi\, x_0x_1x_2x_3x_4 = 0 \right\}$

$G = \left\{ (a_0 \dots a_4) \in \mathbb{Z}/_5^5 \mid \Sigma a_i = 0 \right\} / \left\{ (a,a,a,a,a) \right\} \simeq (\mathbb{Z}/_5)^3$

acts by diagonal mult. by ξ^{a_i}, $\xi = e^{2\pi i/5}$.

$\check{X}_\psi$ = crepant resolution of X_ψ / G (sings. $\overline{C_{ij}} = \{x_i = x_j = 0\} \simeq \mathbb{P}^1$ intersecting at pts P_{ijk}).

has $h^{1,1} = 101$, $h^{2,1} = 1$.

- Note: $(x_0 \dots x_4) \mapsto (\xi^a x_0, x_1 \dots x_4)$ induces $X_\psi \cong X_{\xi\psi}$ hence $\check{X}_\psi \cong \check{X}_{\xi\psi}$

$\Rightarrow$ to get an actual coord. on moduli space, let $z = (5\psi)^{-5}$

★ $z \to 0$ ie. $\psi \to \infty$ corresponds to toric degeneration of X_ψ to $x_0x_1x_2x_3x_4 = 0$ union of 5 coordinate hyperplanes

This is maximally unipotent, and hence a LCSL degeneration!

Need to compute the periods of Ω a holom. volume form on $\check{X}_\psi$.
Because Ω on $\check{X}_\psi$ is induced by G-inv't holom. vol. form on X_ψ,
(quotient by G, then pullback via resolution map $\pi: \check{X}_\psi \to X_\psi/G$)
we can work on X_ψ instead.

$\exists$ Explicit method (Candelas - de la Ossa - Greene - Parkes)
[could also use lots of toric geometry ... see Cox - Katz book].

- 3-cycle β_0 in $\check{X}_\psi$:

for $z=0$, $\{\Pi x_i = 0\} \supset$ tori T^3 in components $\mathbb{P}^3$:

$T_0 = \left\{ (x_0 \dots x_4) \in \mathbb{P}^4 \,/\, x_4 = 1,\ |x_0| = |x_1| = |x_2| = \delta,\ x_3 = 0 \right\}$

for $z \neq 0$: take $x_4 = 1$, $|x_0| = |x_1| = |x_2| = \delta$,
x_3 = the root of $f_\psi = 0$ which tends to 0 as $\psi \to \infty$

(3) namely let $x_3 = (\psi x_0 x_1 x_2)^{1/4}\, y \rightsquigarrow$

$$f_\psi = 0 \iff x_0^5 + x_1^5 + x_2^5 + (\psi x_0 x_1 x_2)^{5/4}\, y^5 + 1 - 5(\psi x_0 x_1 x_2)^{5/4} y = 0$$

$$\text{ie. } \quad y = \frac{y^5}{5} + \frac{1 + x_0^5 + x_1^5 + x_2^5}{(\psi x_0 x_1 x_2)^{5/4}}$$

as $\psi \to \infty$, one root $y \sim \psi^{-5/4}$, the 4 others $\to \sqrt[4]{5}$
$x_3 \sim \psi^{-1}$, the 4 others $\sim \psi^{1/4}$

hence $\exists$ well-defined branch of x_3 that $\to 0$.

This defines a 3-torus T_0 in X_ψ,

G acts on T_0, freely ($T_0 \cap C_{ij} = \phi$) $\rightsquigarrow$ torus $\check{T}_0$ in X_ψ / G and $\check{X}_\psi$

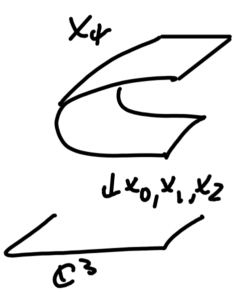
★ Because $\check{T}_0$ still makes sense as smooth submfld for $z=0$, its class $\beta_0 = [\check{T}_0] \in H_3(\check{X}_\psi, \mathbb{Z})$ is preserved by the monodromy.
($\beta_0 \in W_0$ for weight filtration!)

• <u>holom volume form:</u>

in affine subset $x_4 = 1$, let Ω_ψ be 3-form on X_ψ characterized by
$$\Omega_\psi \wedge df_\psi = 5\psi\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \quad \text{at every point of } X_\psi$$

E.g. if we're at a point where $\frac{\partial f_\psi}{\partial x_3} \neq 0$, then can use x_0, x_1, x_2 as local coords on X_ψ & write Ω_ψ as a scalar mult. of $dx_0 \wedge dx_1 \wedge dx_2$.

Necess: $$\Omega_\psi = 5\psi \left.\frac{dx_0 \wedge dx_1 \wedge dx_2}{\partial f_\psi / \partial x_3}\right|_{X_\psi} = 5\psi \left.\frac{dx_0 \wedge dx_1 \wedge dx_2}{(5x_3^4 - 5\psi x_0 x_1 x_2)}\right|_{X_\psi}$$



This doesn't have poles where $\frac{\partial f_\psi}{\partial x_3} = 0$ because there the Jacobian of proj^n to $(x_0 x_1 x_2)$ coords. vanishes, ie. numerator <u>also</u> has a zero.

formulas e.g. in terms of $x_0 x_1 x_3$ etc. still make sense ✓

• There isn't a zero or pole either at $x_4 = 0$: e.g switch to chart $x_3 = 1$ by setting $\tilde{x}_i = \frac{x_i}{x_3}$, then check that Ω_ψ still looks same in new coords.

or: divisor def^d by Ω_ψ is a multiple of $\{x_4 = 0\}$, but linearly ~ 0 since X_ψ Calabi-Yau

(4) * Ω_ψ is G-invariant & induces a holom. volume form on $(X_\psi/G)^{\text{nonsing}}$

pull back via resolution $\rightsquigarrow$ holom. vol. form $\check{\Omega}_\psi$ on $\check{X}_\psi$

(extends across exc. divisors of blowups because resolⁿ is crepant)

* we want to compute $\int_{\check{T}_0} \check{\Omega}_\psi$ or equivtly up to 5^3, $\int_{T_0} \Omega_\psi$

<u>Tool:</u> residue formula $\frac{1}{2\pi i}\int_{S^1} f(z)\,dz = \sum_{\substack{z_i \text{ pole of } f \\ z_i \in D^2}} \operatorname{Res}_{z_i}(f)$

So: let $T^4 = \{ |x_0| = |x_1| = |x_2| = |x_3| = \delta,\ x_4 = 1\} \subset \mathbb{P}^4$:

$$\frac{1}{2\pi i}\int_{T^4} 5\psi \frac{dx_0\,dx_1\,dx_2\,dx_3}{f_\psi} = \int_{T^3_{x_0x_1x_2}} \left(\frac{1}{2\pi i}\int_{S^1_{x_3}} 5\psi \frac{dx_3}{f_\psi}\right) dx_0\,dx_1\,dx_2$$

we've seen only one root of f_ψ near 0

$\Rightarrow$ only one pole, on T_0! residue $= \dfrac{5\psi}{\partial f_\psi/\partial x_3}$

Thus $\ldots = \int_{T_0} \dfrac{5\psi\,dx_0\,dx_1\,dx_2}{\partial f_\psi/\partial x_3} = \int_{T_0} \Omega_\psi$.

$$\text{So } \int_{T_0}\Omega_\psi = \frac{1}{2\pi i}\int_{T^4} \frac{dx_0\,dx_1\,dx_2\,dx_3}{(5\psi)^{-1}(x_0^5+x_1^5+x_2^5+x_3^5+1) - x_0x_1x_2x_3}$$

$$= \frac{-1}{2\pi i}\int_{T^4} \frac{dx_0\,dx_1\,dx_2\,dx_3}{x_0x_1x_2x_3} \frac{1}{1 - \frac{(x_0^5+\cdots+x_3^5+1)}{5\psi\,x_0x_1x_2x_3}}$$

$$= \frac{-1}{2\pi i}\sum_{m=0}^{\infty}\int_{T^4} \frac{dx_0\,dx_1\,dx_2\,dx_3}{x_0x_1x_2x_3} \underbrace{\frac{(x_0^5+\cdots+x_3^5+1)^m}{(5\psi)^m (x_0x_1x_2x_3)^m}}_{(\#)}$$

Now use again residues $\rightsquigarrow$ only terms contributing are terms in (#) not involving any x_i's $\rightsquigarrow$ need term $(x_0^{5n}x_1^{5n}x_2^{5n}x_3^{5n})$ in numerator expansion for $m = 5n$. ie. need to pick each of $x_0^5, \ldots, x_3^5, 1$ n times.

such terms is $(5n)!/(n!)^5$ (e.g. choose x_0's $\binom{5n}{n}$ then x_1's $\binom{4n}{n} \ldots x_3$'s $\binom{2n}{n}$)

Hence $\int_{T_0} \Omega_\psi = -(2\pi i)^3 \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}$

or in terms of $z = (5\psi)^{-5}$, proportional to $\phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$

- This tells us how to scale $\check{\Omega}_\psi$ for normalization.

Then we need another period (along what 3-cycle?) to get canonical coordinate.

Instead of brute-force calc$\stackrel{n}{=}$, use a general fact: all periods $\int_C \check{\Omega}_\psi$ satisfy a same differential equation - the Picard-Fuchs eq$\stackrel{n}{=}$ for the family $\check{X}_\psi$.

- Guess for Picard-Fuchs eqn: $\phi_0(z) = \sum a_n z^n$, $a_n = \frac{(5n)!}{(n!)^5}$

obey recurrence relation $(n+1)^5 a_{n+1} = \frac{(5n+5)!}{(n!)^5} = (5n+5)(\ldots)(5n+1)\, a_n$

or $(n+1)^4 a_{n+1} = 5(5n+1)\ldots(5n+4)\, a_n$

Let $\Theta = z\frac{d}{dz}$: $\Theta\left(\sum c_n z^n\right) = \sum n c_n z^n$ so ϕ_0 solves

$$\Theta^4 \phi = 5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\phi$$

Prop: $[\check{\Omega}_\psi]$ and hence all periods $\int_C \check{\Omega}_\psi$ also satisfy this equation