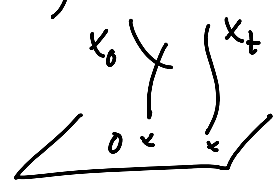
1 <u>Degenerations & monodromy cont'd</u> : (today: linear algebra)

$\mathcal{X} \ni X_t$ family of compact Kähler manifolds,

$\downarrow \quad \downarrow$ $\quad X_t$ smooth, X_0 singular

$D^2 \ni t$



We've seen: <u>monodromy</u> around $t=0$ induces $\varphi_* \in \mathrm{Aut}(H^n(X_{t_0}, \mathbb{Z}))$

<u>Thm:</u>
- all eigenvalues of φ_* are roots of unity.
  Thus $\exists\, N, k$ st. $(\varphi_*^N - Id)^k = 0$
- moreover, $k \le n+1$.

- replacing φ by φ^N ("base change": $X'_t = X_{t^N}$), can assume φ_* is <u>unipotent</u> ie. $(\varphi_* - id)^k = 0$; <u>maximally unipotent</u> := $k = n+1$.

- Can define a <u>weight filtration</u> associated to unipotent φ_*:
  [Comes from Jordan block decomposition of φ_*]

  let $N = \log(\varphi_*) = (\varphi_* - id) - \frac{(\varphi_* - id)^2}{2} + \dots + (-1)^{n+1} \frac{(\varphi_* - id)^n}{n}$

  nilpotent $N^{n+1} = 0$ acting on $V = H^n(X, \mathbb{Q})$

$\Rightarrow \exists$ filtration $0 \subseteq W_0 \subseteq W_1 \subseteq \dots \subseteq W_{2n} = V$ s.t.

$$\begin{cases} N(W_i) \subseteq W_{i-2} \\ N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1} \end{cases}$$

<u>Construction:</u>
- $N^n : W_{2n}/W_{2n-1} \xrightarrow{\sim} W_0$ so: $W_0 := \mathrm{Im}(N^n)$
  $W_{2n-1} := \ker(N^n)$
- then let $V' = W_{2n-1}/W_0$, N induces $N' \in \mathrm{End}(V')$
  (since $W_{2n-1} = \ker N^n \supseteq \mathrm{Im}\, N$ & $W_0 = \mathrm{Im}(N^n) \subseteq \ker N$)

  and $(N')^n = 0 \rightsquigarrow$ by induction,

$$\begin{array}{ccccc} 0 \subseteq W'_0 & \subseteq & \dots & \subseteq & W'_{2n-2} = V' \\ \wr\!\wr & & & & \wr\!\wr \\ W_1/W_0 & \subseteq & \dots & \subseteq & W_{2n-1}/W_0 \end{array}$$

②

and $W_{2n-2} = \{ v \,/\, N^{n-1}(v) \in W_0 = \mathrm{Im}\, N^n \} \supseteq \mathrm{Im}\, N$ so $W_{2n} \xrightarrow{N} W_{2n-2}$

$W_1 = \{ N^{n-1}(v) \,/\, N^n(v) = 0 \} \subseteq \ker N$ so $W_1 \xrightarrow{N} 0$; $W_k \to W_{k-2}$ by induction.

<u>Ex:</u> for the elliptic curve last time, $\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{N}$

$(\varphi - \mathrm{Id})^2 = 0$

$$0 \subseteq W_0 \subseteq W_1 \subseteq W_2 = H^1(C, \mathbb{Q}) \quad (\simeq \mathbb{Q}^2)$$

$\mathrm{Im}\, N = \ker N = \mathrm{span}(a)$ = direction invariant by monodromy.

(<u>Note:</u> if N = Jordan blocks $\underbrace{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}}_{(n+1)\times(n+1)}$ (columns $e_1 \dots e_{n+1}$) then $W_0 = \mathrm{span}(e_1)$ $\quad W_{2n-1} = \mathrm{span}(e_1, \dots e_n)$

reduction to $\underbrace{\begin{pmatrix} 0 & 1 & \\ & \ddots & \\ & & 0 \end{pmatrix}}$ $\quad W_1 = W_0$, $W_{2n-2} = W_{2n-1} \cdots$

$\to W_{2k-2} = W_{2k-1} = \mathrm{span}(e_1, \dots e_k)$. Similar story if $N = \overset{n-1}{\oplus}$ Jordan blocks).

✩ In fact, the interplay of weight filtration with Hodge filtration

$F^p = H^{n,0} \oplus \dots \oplus H^{p,n-p}$ $\quad (H^n = F^0 \supseteq F^1 \supseteq \dots, \quad F^p/F^{p+1} \simeq H^{p,n-p})$

under deform^{s}, griffiths transversality $\Rightarrow \nabla F^p \subseteq F^{p-1}$

gives notion of "mixed Hodge structure".

(Point: $\exists$ limiting Hodge filtration at $t \to 0$ [Schmid])

We won't say more about those.

✩ Now consider a multidimensional family $\begin{matrix} \mathcal{X} \\ \downarrow \\ (D^2)^s \end{matrix}$ smooth over $(D^*)^s$ $\quad (D^* = D^2 - \{0\})$

then we have s monodromies $\varphi_1, \dots, \varphi_s \in \mathrm{Aut}\, H_n(X)$,

$[\varphi_i, \varphi_j] = 0$ $\quad$ (since $\pi_1((D^*)^s) = \mathbb{Z}^s$ abelian)

$\rightsquigarrow$ $N_i = \log \varphi_i$ also commute.

<u>Thm</u> (Cattani-Kaplan)

All elements of the form $\sum \lambda_i N_i$, $\lambda_i > 0$ have the same monodromy weight filtration.

Want to consider a "universal family" of CY near a "deepest corner" :=
"Large complex structure limit point" in moduli space

③ **Def:** (Morrison)

$\mathcal{X} \to (D^*)^s \subset (D^2)^s$ family of CY n-folds, $s = h^{n-1,1}(X)$

s.t. Kodaira-Spencer map $T_t((D^*)^s) \to H^1(TX_t)$

is an isomorphism at every point of $(D^*)^s$

We say $0 \in (D^2)^s$ is a <u>large complex str. limit point</u> (LCSL point)

if (1) the monodromies φ_j around each factor are all unipotent

(2) let $N_j = \log \varphi_j$, and $N = \sum \lambda_j N_j$ for $\lambda_j > 0$ arbitrary.

Then weight filtration $0 \subseteq W_0 \subseteq W_1 \subseteq W_2 \subseteq \dots \subseteq W_{2n} = H^n(X, \mathbb{Q})$

has $\dim W_0 = \dim W_1 = 1$, $\dim W_2 = \dim W_3 = s+1$.

(3) let α_0^* generator of W_0, $\alpha_1^* \dots \alpha_s^* \in W_2$ s.t. $(\alpha_0^* \dots \alpha_s^*)$ basis

Then $\exists\, m_{jk} \in \mathbb{Q}$ st. $N_j(\alpha_k^*) = m_{jk}\alpha_0^*$ (ie. $\varphi_j(\alpha_k^*) = \alpha_k^* + m_{jk}\alpha_0^*$)

Require: (m_{jk}) invertible matrix.

This essentially says: → family is locally "full" deformation

→ $W_0 = W_1$ rank 1 := singles out 1-dim'l subspace of $H^n(X)$ span(α_0^*) preserved by the whole monodromy.

→ W_2 dim. s & invertibility means: for each factor D^2 we get a class $\tilde{\alpha}_j^*$ st. $\varphi_j(\tilde{\alpha}_j^*) = \tilde{\alpha}_j^* + \alpha_0^*$ & $\tilde{\alpha}_j^*$ inv't under other φ_i

Fact: if $h^{n-1,1} = s = 1$ then this is equivalent to: monodromy around 0 is maximally unipotent

Ex: family of elliptic curves seen last time is a LCSL point.

* Now, for a family of CY 3-folds:

by def^n, $0 \subset \underbrace{W_0 = W_1}_{\text{dim. } 1} \subset \underbrace{W_2 = W_3}_{\text{dim } s+1 = h^{2,1}+1} \subset \underbrace{W_4 = W_5}_{\text{dim. } 2s+1} \subset \underbrace{W_6 = H^3(X,\mathbb{Q})}_{\text{dim. } 2s+2}$

use $N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1}$ to get dim. W_3, W_4, W_5

④

* $H^3(X)$ carries intersection pairing $(\cdot,\cdot)$, preserved by φ_*
  $\Rightarrow$ $N = \log \varphi_*$ is in the Lie algebra: $(x, Ny) + (Nx, y) = 0$.

<u>Lemma:</u> ‖ $W_{4-2i} = W_{2i}^{\perp}$

<u>Pf:</u>
- $\left.\begin{array}{l} W_0 = \operatorname{Im} N^3 \\ W_4 = W_5 = \ker N^3 \end{array}\right\} \Rightarrow$ if $x \in W_4$, $N^3 y \in W_0$ then $(x, N^3 y) = -(N^3 x, y) = 0$.
  \+ dimensions match
- $N(W_4) = W_2$ (onto since: $N : W_4/W_3 = W_2 \xrightarrow{\sim} W_2/W_1 = W_0$
  and: $W_0 = \operatorname{Im} N^3 = N(\underbrace{\operatorname{Im} N^2}_{\subset W_4})$)
  $\Rightarrow$ if $x, Ny \in W_2$ ($y \in W_4$) then
  $(x, Ny) = -(Nx, y) = 0$ (since $W_0 \perp W_4$)
  \+ dims match.

∎

* Passing to $H_3(X, \mathbb{Q})$ by Poincaré duality, let $S_i =$ P.D. of W_i
  or equivalently, viewing $H_3 \simeq (H^3)^*$, $S_i =$ annihilator of W_{4-2i}

<u>Prop:</u> ‖ Given LCSL point in moduli space of CY 3-folds w/ $h^{2,1} = s$,
‖ $\exists$ $\mathbb{Z}$-basis $\alpha_0, \dots \alpha_s, \beta_0, \dots, \beta_s$ of $H_3(X)$ s.t.
‖ $\beta_0 \in S_0$, $\beta_1 \dots \beta_s \in S_2$, $\alpha_1 \dots \alpha_s \in S_4$, $\alpha_0 \in S_6 = H_3(X)$
‖ s.t. $(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0$, $(\alpha_i, \beta_j) = \delta_{ij}$.

<u>Pf:</u>
- $\beta_0 :=$ $\mathbb{Z}$-generator of S_0 (unique up to sign)
- extend to a $\mathbb{Z}$-basis of S_2;
  by lemma, S_2 is Lagrangian wrt $(\cdot,\cdot)$ so $(\beta_i, \beta_j) = 0$
- let $\beta_i^* =$ dual basis of $S_2^* = H^3/W_2$ ie. $\beta_i^*(\beta_j) = \delta_{ij}$
  $\alpha_i \in H_3 =$ Poincaré dual of some lift of β_i^* to H^3 $\Rightarrow$ $(\alpha_i, \beta_j) = \delta_{ij}$
  can ensure $(\alpha_i, \alpha_j) = 0$ inductively by $\alpha_i \leftarrow \alpha_i - \sum (\alpha_i, \alpha_j) \beta_j$
- $\alpha_1 \dots \alpha_s \in S_4$ since $(\alpha_i, \beta_0) = 0 \Rightarrow \alpha_i \in S_0^{\perp}$

∎

(5)

<u>Canonical coordinates</u>: given $\mathcal{X} \to (\mathbb{D}^*)^s$ LCSL,

let $\Omega(t_1 \ldots t_s)$ = holom. vol. form on $X_{(t_1 \ldots t_s)}$, normalized so that

$$\int_{\beta_0} \Omega(t_1 \ldots t_s) = 1 .$$

Then $w_i(t_1 \ldots t_s) := \int_{\beta_i} \Omega(t_1 \ldots t_s)$

Not quite a coordinate because of monodromy:

as t_j goes around the origin, $\beta_i \longmapsto \varphi_j(\beta_i) = \beta_i - m_{ji}\beta_0$

for some $m_{ji} \in \mathbb{Z}$

(integer since $\mathbb{Z}$-classes!)

(in fact $m_{ji} = m_{ij}$ from def^n of LCSL)

so $w_i \longmapsto w_i - m_{ji}$

instead set $q_i = \exp(2\pi i w_i)$ well-defined functions on $(\mathbb{D}^*)^s$

<u>canonical coordinates</u> (canonical only once basis $\{\beta_i\}$ is chosen!)

(q_i has $\underset{\text{pole}}{\text{zero}}$ of order $\underset{m_{ji}}{(-m_{ji})}$ along $t_j = 0$)

if m_{ji}'s are nonpositive then get coords. on $(D^2)^s$ ...

choose basis of S_2 appropriately!

<u>Ex</u>: for elliptic curves of Tuesday, $q = \exp(2\pi i \tau(t))$, $\tau(t) = \int_b \Omega$

where $\int_a \Omega = 1$.

Last time, saw e_i basis of $H^2(\check{X}, \mathbb{Z})$, $e_i \in$ Kähler cone

$\rightsquigarrow$ coords. on complexified Kähler moduli space:

if $[B + i\omega] = \sum \check{t}_i e_i$, let $\check{q}_i = \exp(2\pi i \check{t}_i) \in \mathbb{C}^*$

(ie. $\check{t}_i = \int_{e_i^*} B + i\omega$)

<u>Ex</u>: for T^2, $\check{q}_i = \exp(2\pi i \cdot \int_{T^2} B + i\omega)$

⑥

<u>Mirror symmetry statement:</u>

$f: \mathcal{X} \to (D^*)^s$ family of CY 3-folds with LCSL point at 0.

Then $\exists$ CY 3-fold $\check{X}$ + $\exists$ choice of bases $\alpha_0 \dots \alpha_s, \beta_0 \dots \beta_s$ on $H_3(X)$

$e_1 \dots e_s$ on $H^2(\check{X})$

s.t. under the map $m: (D^*)^s \to \mathcal{M}_{K\ddot{a}h}(\check{X})$

$(q_1 \dots q_s) \longmapsto (\check{q}_1 \dots \check{q}_s)$ in canonical coordinates

the Yukawa couplings $\left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right\rangle_p = \left\langle \frac{\partial}{\partial \check{q}_1}, \frac{\partial}{\partial \check{q}_2}, \frac{\partial}{\partial \check{q}_3} \right\rangle_{m(p)}$

(2,1) Yukawa coupling: $\int_X \Omega \wedge \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} \Omega$

at point p given by pseudo q_i

(1,1) Yukawa coupling

ie. GW invts.

$2\pi i \, \check{q}_i \frac{\partial}{\partial \check{q}_i} = \frac{\partial}{\partial \check{t}_i} = e_i \in H^2(\check{X}, \mathbb{Z})$

<u>Next time</u>: Computing canonical coords & (2,1)-coupling on mirror quintics.