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The quintic 3-fold & its mirror

Simplest Calabi-Yaus = hypersurfaces in toric varieties, esp:

X = smooth hypersurface in $\mathbb{CP}^{n+1}$ defined by a polynomial P of deg. $d = n+2$:
ie. section of $\mathcal{O}_{\mathbb{P}^{n+1}}(d)$

Smoothness $\Rightarrow$
$$NX \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^{n+1}}(d)_{|X}$$
$$v \longmapsto \nabla_v P \ (= dP(v))$$

so $T\mathbb{P}^{n+1}{}_{|X} = TX \oplus NX = TX \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(d)_{|X}$ "adjunction"

Passing to dual & determinant: $\Omega^{n+1}_{\mathbb{P}^{n+1}}{}_{|X} \cong \Omega^n_X \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-d)_{|X}$

Now: $T_\ell \mathbb{P}^{n+1} \oplus \mathbb{C} = \mathrm{Hom}(\ell, \ell^\perp) \oplus \mathrm{Hom}(\ell,\ell) = \mathrm{Hom}(\ell, \mathbb{C}^{n+2}) = \mathrm{Hom}(\mathcal{O}(-1)_\ell, \mathbb{C}^{n+2})$

$\Rightarrow T\mathbb{P}^{n+1} \oplus \mathcal{O} \simeq \mathcal{O}(1)^{\oplus n+2}$

& passing to dual & det., $\Omega^{n+1}_{\mathbb{P}^{n+1}} \otimes \mathcal{O} \cong \mathcal{O}(-1)^{\otimes(n+2)} = \mathcal{O}(-(n+2))$

Get: $\mathcal{O}_{\mathbb{P}^{n+1}}(-(n+2))_{|X} \cong \Omega^n_X \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-d)_{|X}$

$\Rightarrow \Omega^n_X \cong \mathcal{O}$ if $d = n+2$.

Ex: cubic curve in $\mathbb{P}^2$ = elliptic curve (genus 1, T^2)
quartic surface in $\mathbb{P}^3$ = K3 surface
quintic in $\mathbb{P}^4$ ← the world's most studied CY 3-fold

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* Cohomology of the quintic: by Lefschetz hyperplane theorem,
inclusion induces $i_*: H_r(X) \xrightarrow{\sim} H_r(\mathbb{CP}^4)$ for $r < n = 3$
so $H_1(X) = 0$, $H_2(X) = H_2(\mathbb{CP}^4) = \mathbb{Z}$
$\rightarrow h^{1,0} = 0 \rightarrow h^{2,0} = 0$ by argument seen before $h^{1,1} = 1$

* $\chi(X) = e(TX).[X] = c_3(TX).[X]$.

By working out $c(T\mathbb{P}^4)_{|X} = c(TX)\, c(\mathcal{O}_{\mathbb{P}^4}(5))_{|X}$ (from adjunction)

$c(T\mathbb{P}^4) = c(T\mathbb{P}^4 \oplus \mathcal{O}) = c(\mathcal{O}(1)^{\oplus 5}) = (1+h)^5$

($h = c_1(\mathcal{O}(1))$ = generator of $H_2(\mathbb{CP}^4)$ = P.D. to hyperplane)

$$\rightsquigarrow \quad (1+h_{|X})^5 = 1+5h_{|X}+10h^2_{|X}+10h^3_{|X} = (1+c_1+c_2+c_3)(1+5h_{|X})$$

gives $\quad c_1 = 0, \quad c_2 = 10h^2_{|X}, \quad c_3 = -40h^3_{|X}$

so $\chi(X) = -40\, h^3 \cdot [X] = -40\,([\text{line}] \cap [X]) = -40 \cdot 5 = -200$

conclude : $\underset{H_0}{1} + \underset{H_2}{1} - \dim H_3(X) + \underset{H_4}{1} + \underset{H_6}{1} = -200 \Rightarrow \dim H_3 = 204$

Since $h^{3,0} = h^{0,3} = 1$, get $h^{1,2} = h^{2,1} = 101$.

In fact, • $h^{1,1} = 1$ sympl. parameter := area of generator of $H_2(X)$ $(= [\text{line}] \in H_2(\mathbb{P}^4))$

• $h^{2,1} = 101$ complex parameters:

equation of quintic : $h^0(\mathcal{O}_{\mathbb{P}^4}(5)) = \binom{9}{5} = 126$

$\rightsquigarrow$ zero sets $\dim \mathbb{P}H^0(\mathcal{O}(5)) = 125$

minus $\mathrm{Aut}(\mathbb{CP}^4) = PGL(5,\mathbb{C}) = \dim. 24$ $\Big)$ $125 - 24 = 101$

ie. all $\mathbb{C}$ deform$^{\text{s}}$ are still quintics.

* <u>Mirror</u>: start with distinguished family of quintic 3-folds

$$X_\psi = \{(x_0:\dots:x_4) \in \mathbb{P}^4 \,/\, f_\psi = x_0^5 + \dots + x_4^5 - 5\psi\, x_0x_1x_2x_3x_4 = 0\}$$

Let $G = \{(a_0 \dots a_4) \in (\mathbb{Z}/5)^5 \,/\, \Sigma a_i = 0\} / (\mathbb{Z}/5) = \{(a,a,a,a,a)\}$

$(G \cong (\mathbb{Z}/5)^3)$

Then G acts on X_ψ by $(x_j) \longmapsto (x_j \xi^{a_j})$ where $\xi = e^{2\pi i/5}$

$\Big($ • $\Sigma a_j = 0 \bmod 5 \Rightarrow f_\psi$ is G-invariant
• x_j are homogeneous coords. $\Rightarrow (1,1,1,1,1)$ acts trivially $\Big)$

X_ψ is smooth for ψ generic $(\psi^5 \neq 1)$, but X_ψ/G singular !!

Fixed points $=$ $(x_0:\dots:x_4) \in X_\psi$ s.t. at least two coords. are 0.

This consists of:

• curve $C_{01} = \{x_0 = x_1 = 0,\ x_2^5 + x_3^5 + x_4^5 = 0\}$ stabilizer $\mathbb{Z}/5 = \{(a,-a,0,0,0)\}$

$C_{01}/G \simeq \mathbb{P}^1$ (line $y_2 + y_3 + y_4 = 0$ in $\mathbb{P}^2$, $y_i = x_i^5$)

one of 10 C_{ij}'s

• point $\in P_{012} = \{x_0 = x_1 = x_2 = 0,\ x_3^5 + x_4^5 = 0\}$ stabilizer $(\mathbb{Z}/5)^2$

5 pts, $P_{012}/G = \{pt\}$

10 such P_{ijk}'s.

$\Rightarrow$ singular locus of X_ψ/G = 10 curves $\bar{C}_{ij} = C_{ij}/G$ $(\simeq \mathbb{P}^1)$
with $\bar{C}_{ij}, \bar{C}_{jk}, \bar{C}_{ik}$ meeting at a point $\bar{P}_{ijk}$.

• $\check{X}_\psi :=$ resolution of singularities of (X_ψ/G)

(ie. $\check{X}_\psi$ smooth, $\check{X}_\psi \xrightarrow{\pi} X_\psi/G$, π isom. outside $\pi^{-1}(\cup C_{ij})$)

Explicit constrⁿ complicated; can use toric geometry

Can show: This is a crepant resolution (canonical bundle $K_{\check{X}_\psi} = \pi^* K_{X_\psi/G}$)

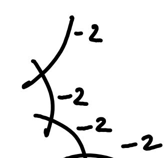
$\rightsquigarrow$ CY condition is preserved, $\check{X}_\psi$ is a CY 3-fold.

Ex: along $\bar{C}_{ij}$ (away from $\bar{P}_{ijk}$), X_ψ/G looks like $(\mathbb{C}^2/\mathbb{Z}_5) \times \mathbb{C}$,

$(x_1, x_2, x_3) \sim (\xi^a x_1, \xi^{-a} x_2, x_3)$

$$\mathbb{C}^2/(\mathbb{Z}/5) \simeq \{uv = w^5\} \subset \mathbb{C}^3$$
$$[x_1, x_2] \longmapsto [x_1^5, x_2^5, x_1 x_2]$$

A_4-singularity



can resolve by blowing up twice $\rightsquigarrow$ get 4 exc. divisors

(this is crepant!)

Do this for each $\bar{C}_{ij}$ $\rightsquigarrow$ create 40 divisors

Also, resolution at each $\bar{P}_{ijk}$ creates 6 divisors $\times 10 = 60$

$\rightsquigarrow$ $\check{X}_\psi$ contains 100 new divisors besides hyperplane section ...

can in fact show that, indeed, $h^{1,1}(\check{X}_\psi) = 101$

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Can also show $h^{2,1}(\check{X}_\psi) = 1$

(consistent with: we were only able to build a 1-param. family)

So Hodge diamonds match.

quintic

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\check{X}_\psi$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We want to see how mirr symm. predicts the GW invariants N_d ("# rat'l curves" n_d) of quintic.

For that, need to understand:

- mirror map between
$$\begin{cases} \text{Kähler param. } q = \exp\left(2\pi i \int_{\text{line}} B + i\omega\right) \text{ on quintic} \\ \text{Complex param. } \psi \text{ on mirror } \check{X}_\psi \end{cases}$$
(will also give, by differentiating, isom. $H^{1,1}(\text{quintic}) \xrightarrow{\sim} H^{2,1}(\check{X}_\psi)$)
- calc^n of Yukawa coupling on $H^{2,1}(\check{X}_\psi)$

Degenerations and the mirror map

Last time, saw e_i basis of $H^2(X, \mathbb{Z})$, $e_i \in$ Kähler cone

$\rightsquigarrow$ coords. on complexified Kähler moduli space:

if $[B + i\omega] = \sum t_i e_i$, let $q_i = \exp(2\pi i t_i) \in \mathbb{C}^*$

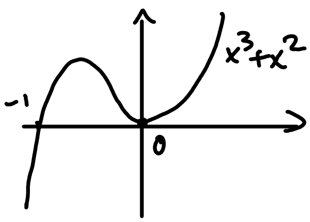
$q_i \to 0$ corresponds to large volume limit ($\mathrm{Im}(t_i) \to \infty$)

Physics predicts that the mirror situation = degeneration to "large cx. structure limit" and that, near such a limit point, $\exists$ "canonical coordinates" on cx. moduli space – making it possible to describe the mirror map.

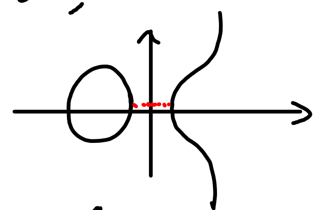
- <u>Degeneration</u> := family $\begin{array}{ccc} \mathcal{X} & \supset & X_t \\ \pi\downarrow & & \downarrow \\ D^2 & \ni & t \end{array}$ ($\mathcal{X}$ ← complex manifold) where for $t \neq 0$, $X_t \underset{\text{diff.}}{\cong} X$ (with varying J)
  for $t=0$, X_0 typically singular

<u>Ex:</u> elliptic curves $C_t = \{y^2 z = x^3 + x^2 z - t z^3\} \subset \mathbb{CP}^2$

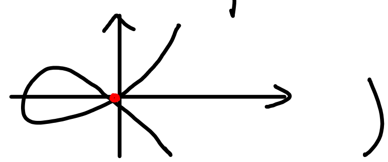
(in affine coords. C_t: $y^2 = x^3 + x^2 - t$,



real part for $t>0$:

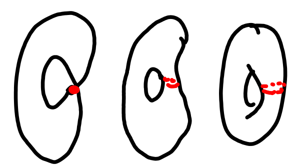


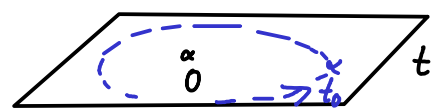
for $t=0$:



)

C_t smooth for $t \neq 0$, C_0 <u>nodal</u> : actually



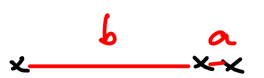


- <u>Monodromy</u>: follow family (X_t) as t varies along loop in $\pi_1(D^2 \setminus \{0\}, t_0)$ going around origin. All X_t's diffeomorphic, so induces a monodromy diffeomorphism φ of X_{t_0}, defined up to isotopy.
  Induces: $\varphi_* \in \mathrm{Aut}(H_n(X_{t_0}, \mathbb{Z}))$

In above example: φ acts on $H_1(C_{t_0}) = \mathbb{Z}^2$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (Dehn twist)

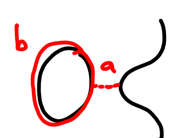
(<u>observe</u>: $C_t \xrightarrow{2:1} \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by projection to x
branch pts are $\begin{cases} \text{roots of } x^3+x^2-t \\ \infty \end{cases}$

Near $t \to 0$:



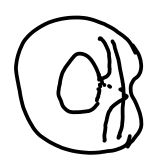
not near −1   2 roots near 0

$\Longleftrightarrow$



when t goes around 0, roots near 0 rotate

, induces



⇒ monodromy maps to

The complex parameter t is ad hoc. A more natural way to describe the degeneration would be to describe C_t as an abstract elliptic curve $C_t \simeq \mathbb{C}/\mathbb{Z}+\tau(t)\mathbb{Z}$

then $\tau(t)$, or rather $\exp(2\pi i \tau)$ it turns out, is a better quantity...

Equip C_t with a holom. vol. form Ω_t normalized so $\int_a \Omega_t = 1 \ \forall t$

Then let $\tau(t) = \int_b \Omega_t$ ... as t goes around origin, $\underline{\tau(t) \to \tau(t)+1}$!!

since $b \mapsto b+a$

still, $q(t) = e^{2\pi i \tau(t)}$ is single-valued

As $t \to 0$, $\text{Im}\,\tau(t) \to \infty$ and $q(t) \to 0$

for $t \in \mathbb{R}_+$, $t \to 0$:

$\int_a \frac{dx}{y} \in -i\mathbb{R}^+$ and $\to 0$

$\int_b \frac{dx}{y} \in \mathbb{R}^+$ and $\to$ constant

ratio $\to +i\infty$

$q(t)$ holom. function of t, and goes around 0 once when t does ie. has a single root at $t=0$.

$\Rightarrow q$ is a local coordinate for family!

<u>Next time</u>: analogue of this for a family of Calabi-Yau 3-folds?