1 <u>Recall:</u> (X,ω) symplectic manifold, J compatible a.c.s., $\beta \in H_2(X,\mathbb{Z})$

Then: $\overline{\mathcal{M}}_{g,k}(X,J,\beta) = \left\{ \begin{array}{l} \text{(possibly nodal) } J\text{-hol. curves of genus } g \\ \text{w/ } k \text{ marked pts, representing class } \beta \end{array} \right\} / \sim$

might not be nice, but carries a fundamental class $[\ ] \in H_{2d}(\overline{\mathcal{M}}_{g,k}(X,J,\beta), \mathbb{Q})$

where $2d = 2\langle c_1(TX), \beta\rangle + 2(n-3)(1-g) + 2k$

also evaluation maps $ev_1, \dots, ev_k : \overline{\mathcal{M}}_{g,k}(X,J,\beta) \longrightarrow X$

$$(\Sigma, j, z_1 \dots z_k, u) \longmapsto u(z_i)$$

$\rightarrow$ <u>Gromov-Witten invariants</u>: given $\alpha_1, \dots, \alpha_k \in H^*(X)$, $\Sigma \deg(\alpha_i) = 2d$,

$$\langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,k}(X,J,\beta)]} ev_1^*\alpha_1 \wedge \dots \wedge ev_k^*\alpha_k$$

or equivalently by Poincaré duality, given cycles $C_1, \dots C_k$ $([C_i] = PD(\alpha_i))$ and assuming the evaluation maps are transverse to them, this is

$$\#(ev_1^{-1}(C_1) \cap \dots \cap ev_k^{-1}(C_k) \cap [\dots]) = \#(ev_*[\overline{\mathcal{M}}_{g,k}(X,J,\beta)] \cap (C_1 \times \dots \times C_k))$$

ie. "count of genus g holom. curves in class β passing through $C_1, \dots, C_k$".

In general $\in \underline{\underline{\mathbb{Q}}}$!

- In the case of a Calabi-Yau 3-fold, dim. formula simplifies to

$\dim \overline{\mathcal{M}}_{g,k}(X,J,\beta) = 2k$, ie. holomorphic curves are <u>isolated</u>.

We're interested in genus 0, $k=3$ marked points: $\mathcal{M}_{0,3} = \{pt\}$, namely $\Sigma = (S^2, j_0, \{0,1,\infty\})$. Then for $\Sigma \deg \alpha_i = 6$,

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} = \int_{[\overline{\mathcal{M}}_{0,3}(X,J,\beta)]} ev_1^*\alpha_1 \wedge ev_2^*\alpha_2 \wedge ev_3^*\alpha_3$$

If $\deg \alpha_i = 2$, taking dual cycles C_i (of codim. 2):

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} = \#\left\{ u: S^2 \to X \ J\text{-holom., class } \beta \middle/ \begin{array}{l} u(0) \in C_1 \\ u(1) \in C_2 \\ u(\infty) \in C_3 \end{array} \right\} / \sim$$

$$= (C_1 \cdot \beta)(C_2 \cdot \beta)(C_3 \cdot \beta) \ \underbrace{\#\{u: S^2 \to X \ J\text{-hol. class } \beta\}/\sim}_{N_\beta = \#[\overline{\mathcal{M}}_{0,0}(X,J,\beta)]}$$

reparametrization acts transitively on triples of pts.
$(C_i \cdot \beta)$ pts of S^2 are mapped to C_i (counting w/ multiplicity)

② In other terms: $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} = \left(\int_\beta \alpha_1\right)\left(\int_\beta \alpha_2\right)\left(\int_\beta \alpha_3\right) . \# [\bar{\mathcal{M}}_{0,0}(X,J,\beta)]$.

(perhaps easier to see directly... integrating over part of $\bar{\mathcal{M}}_{0,3}(\dots)$ that corresponds to a fixed rational curve w/ different positions of marked pts)

Except ... if $\beta = 0$, then constant maps only $\Rightarrow$ need $u(0) = u(1) = u(\infty) \in C_1 \cap C_2 \cap C_3$

ie. $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,0} = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3$.

- Yukawa Coupling: physicists write (complexified Kähler class $B + i\omega$)

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\substack{\beta \in H_2(X) \\ \beta \neq 0}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} \, e^{2\pi i \int_\beta B + i\omega}$$

Better: treat this as a formal power series

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\substack{\beta \in H_2(X) \\ \beta \neq 0}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} \, q^\beta$$

$\in \Lambda :=$ completion of group ring $\mathbb{Q}[H_2(X)] = \left\{ \sum_{\text{finite}} a_{\beta_i} q^{\beta_i} \right\}$

(allowing infinite sums if $\int_{\beta_i} \omega \to +\infty$)

Rmk: another way to encode same data is as a product structure on cohomology of X: namely, fix (η_i) basis of $H^*(X)$, and let (η^i) dual basis ie. $\int_X \eta_i \wedge \eta^j = \delta_{ij}$. Then set

$$\alpha_1 * \alpha_2 = \sum_i \langle \alpha_1, \alpha_2, \eta^i \rangle \eta_i = \alpha_1 \wedge \alpha_2 + \sum_{\beta \neq 0, i} \langle \alpha_1, \alpha_2, \eta^i \rangle_{0,\beta} \, q^\beta \eta_i$$

Def/Thm: quantum cohomology: $QH^*(X) = (H^*(X;\Lambda), *)$ associative algebra

* Can view q as a set of coordinates on the complexified Kähler moduli space.

(X,J) complex: Kähler cone $K(X,J) = \{[\omega] / \omega \text{ Kähler}\} \subset H^{1,1}(X) \cap H^2(X,\mathbb{R})$

This is an open convex cone: (- nondegeneracy is an open condition
- convex combinations of Kähler forms are Kähler)

$\dim_{\mathbb{R}} K(X,J) = h^{1,1}(X)$ ... but becomes a $\mathbb{C}$ manifold by adding "B-field"

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Def: (X,J) CY 3-fold with $h^{1,0}=0$ (so $h^{2,0}=0$, and $H^{1,1}=H^2$)
$\Rightarrow$ the complexified Kähler moduli space:
$$\mathcal{M}_{\text{käh}}(X) := \left(H^2(X,\mathbb{R}) + i\,K(X,J)\right)/H^2(X,\mathbb{Z})$$
$$= \{[B+i\omega] \,/\, \omega \text{ kähler}\}/H^2(X,\mathbb{Z}).$$

Choose $(e_i)_{i=1..m}$ basis of $H^2(X,\mathbb{Z})$, $e_1 \dots e_m \in \overline{K(X,J)}$
(exists by openness)

Write $[B+i\omega] = \sum t_i e_i$, $t_i \in \mathbb{C}/\mathbb{Z}$

Then coordinates on $\mathcal{M}_K := \{ q_i = \exp(2\pi i t_i)\} \in$ open subset of $(\mathbb{C}^*)^m$ containing $(\mathbb{D}^*)^m$

and $q^\beta \longleftrightarrow q_1^{d_1} \cdots q_m^{d_m}$, where $d_i = \int_\beta e_i$ positive integers
(since e_i kähler, $\beta = [\text{cx. curve}]$)

Remark: GW invts vs. enumerative geometry:

Let $N_\beta = \#[\mathcal{M}_{0,0}(X,J,\beta)] \in \mathbb{Q}$, then we've seen that

$$\alpha_1,\alpha_2,\alpha_3 \in H^2(X) \rightsquigarrow \langle \alpha_1,\alpha_2,\alpha_3\rangle = \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 + \sum_{\beta\neq 0} \langle\alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta}\, q^\beta$$
$$= \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 + \sum_{\beta\neq 0} \left(\int_\beta \alpha_1\right)\left(\int_\beta \alpha_2\right)\left(\int_\beta \alpha_3\right) N_\beta\, q^\beta$$

Yet the first day I wrote $\sum_{\beta\neq 0} \cdots\cdots n_\beta \dfrac{q^\beta}{1-q^\beta}$ instead

where "n_β = # rational curves in class β"?

Discrepancy is due to expected contributions of <u>multiple covers</u>.

- let $C \subset X$ Calabi-Yau 3-fold be an embedded rational curve ($C \simeq \mathbb{P}^1$),

  By a thm of Grothendieck, any holom. vect. bundle $/\mathbb{P}^1$ splits as direct sum of line bundles: so $NC = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$

  ($\mathcal{O}(d)$ = sections are deg. d homogeneous holom. functions on $\mathbb{C}^2$)
  $\mathcal{O}(-1)$ = tautological line bundle

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$$c_1(TX)\cdot[C] = 0 = c_1(TC)\cdot[C] + c_1(NC)\cdot[C] = 2+d_1+d_2$$

$\Rightarrow d_1+d_2 = -2$, "generic case" is $NC = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$

Then C is automatically regular as a J-hol. curve.

$\rightsquigarrow$ C contributes 1 to $N_{[C]}$.

Q: contrib^{n} of mult. covers of C to the GW-invariant $N_{[kC]}$?

$\mathcal{M}(kC) \subset \mathcal{M}_{0,0}(X,J,k[C])$ component consisting of covers of C (has excess dimension) $\Rightarrow \#[\mathcal{M}(kC)]^{vir}$?

Thm: || If $NC = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$ then the contribution of C to $N_{k[C]}$ is $1/k^3$

There are various proofs, none particularly easy. E.g.: Voisin shows $\exists$ perturbation of $\bar\partial$-equations, $\bar\partial_J u = \nu(z,u(z))$ s.t. moduli space $\tilde{\mathcal{M}}_3(kC)$ of perturbed J-hol. maps with 3 marked pts, representing $k[C]$, & whose image $\subset$ small neighborhood of C, is smooth of $\dim_{\mathbb{R}} = 6$; moreover, $(ev_1\times ev_2\times ev_3)_*[\tilde{\mathcal{M}}_3(kC)] = [C\times C\times C] \in H_6(X\times X\times X)$.

- So contribution of C to $\langle\alpha_1,\alpha_2,\alpha_3\rangle_{0,k[C]}$ is

$$\int_{ev_*[\tilde{\mathcal{M}}_3(kC)]} \alpha_1\times\alpha_2\times\alpha_3 = \int_{[C]}\alpha_1\int_{[C]}\alpha_2\int_{[C]}\alpha_3$$
$$= \frac{1}{k^3}\int_{[kC]}\alpha_1\int_{[kC]}\alpha_2\int_{[kC]}\alpha_3.$$

Hence, expect: || $$N_\beta = \sum_{\beta=k\gamma}\frac{1}{k^3}n_\gamma \quad (*)$$

and now, $$\sum_\beta \left(\int_\beta\alpha_1\right)\left(\int_\beta\alpha_2\right)\left(\int_\beta\alpha_3\right) N_\beta\, q^\beta$$
$$= \sum_{\gamma;\,k\geq 1}\left(\int_{k\gamma}\alpha_1\right)\left(\int_{k\gamma}\alpha_2\right)\left(\int_{k\gamma}\alpha_3\right)\frac{n_\gamma}{k^3}q^{k\gamma}$$
$$= \sum_\gamma \left(\int_\gamma\alpha_1\right)\left(\int_\gamma\alpha_2\right)\left(\int_\gamma\alpha_3\right) n_\gamma \underbrace{\sum_{k\geq 1} q^{k\gamma}}_{=\frac{q^\gamma}{1-q^\gamma}}$$

⑤ However, how to define "n_γ = # curves in class γ", and whether those numbers satisfy $(*)$, is unclear, or at least, outside the scope of this class. See: $\left\{\begin{array}{l}\text{Gopakumar-Vafa conj.}\\ \text{Donaldson-Thomas invts,}\\ \text{MNOP conjecture}\end{array}\right.$

Instead: take $(*)$ as a <u>definition</u> of n_γ

and hope these might be integers & actual curve counts ....