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Lecture 4 – Pseudoholomorphic curves: Review of last time:

(X^{2n},ω) symplectic manifold, J compatible almost-$\mathbb{C}$ structure

(Σ, j) Riemann surface of genus g, $z_1 \dots z_k \in \Sigma$ marked points

Moduli space $\mathcal{M}_{g,k} = \{(\Sigma, j, z_1, \dots z_k)\}/_{\text{biholomorphism}}$ $\quad (\dim_{\mathbb{C}} = 3g-3+k)$

Def: $u: \Sigma \to X$ is <u>J-holomorphic</u> if $J \circ du = du \circ j$

ie. $\bar\partial_J u = \frac{1}{2}(du + J\, du\, j) = 0 \in \Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes u^* TX)$

Def: $\mathcal{M}_{g,k}(X,J,\beta) = \left\{ (\Sigma, j, z_1, \dots z_k),\ u: \Sigma \to X \,/\, \begin{matrix} u_*[\Sigma] = \beta \\ \bar\partial_J u = 0 \end{matrix} \right\} /\sim$

$\beta \in H_2(X)$

(equivalence relation: $\phi: \Sigma \xrightarrow{\sim} \Sigma'$, $\phi(z_i) = z_i'$, $\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[d, "\phi"'] & X \\ \Sigma' \arrow[ur, "u'"'] & \end{tikzcd}$)

The linearized operator $D_{\bar\partial}: W^{k+1,p}(\Sigma, u^*TX) \times T\mathcal{M}_{g,k} \to W^{k,p}(\Sigma, \Omega^{0,1} \otimes u^*TX)$

$$D_{\bar\partial}(v, j') = \bar\partial v + \tfrac{1}{2} \nabla_v J \circ du \circ j + \tfrac{1}{2} J \circ du \circ j'$$

is Fredholm, of $\text{index}_{\mathbb{R}} = 2d := 2\langle c_1(TX), \beta \rangle + n(2-2g) + \underbrace{(6g-6+2k)}_{\dim \mathcal{M}_{g,k}}$

- <u>Transversality:</u> say u is <u>regular</u> if $D_{\bar\partial}$ onto at u.

  (then $\mathcal{M}_{g,k}(X,J,\beta)$ smooth at u)

Def: $u: \Sigma \to X$ is <u>simple</u> ("somewhere injective") if $\exists\, z \in \Sigma$ s.t. $\begin{cases} du(z) \text{ injective} \\ u^{-1}(u(z)) = \{z\} \end{cases}$

otherwise, u factors through a covering $\Sigma \to \Sigma' \to X$

$\mathcal{M}^*_{g,k}(X,J,\beta) = \{\text{simple } J\text{-hol. curves}\}$

Thm: $\mathcal{J}^{reg}(X,\beta) = \{ J \in \mathcal{J}(X,\omega) / \text{ every simple } J\text{-hol. curve in class } \beta \text{ is regular}\}$

is a Baire subset in $\mathcal{J}(X,\omega)$

For $J \in \mathcal{J}^{reg}(X,\beta)$, $\mathcal{M}^*_{g,k}(X,J,\beta)$ is smooth of real dim. $2d$ and carries a natural orientation.

* <u>Moreover:</u> $\forall\, J_0, J_1 \in \mathcal{J}^{reg}(X,\beta)$, $\exists$ (dense set of choices of) path $\{J_t\}_{t\in[0,1]}$ s.t.

$\coprod_{t\in[0,1]} \mathcal{M}^*_{g,k}(X,J_t,\beta)$ smooth cobordism between $\mathcal{M}^*_{g,k}(X,J_0,\beta)$ and $\mathcal{M}^*_{g,k}(X,J_1,\beta)$

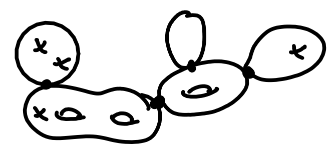
but we need a <u>compactness</u> result, else "# curves" not indep't of $J \in \mathcal{J}_{reg}$!

2 * So far we haven't really used the symplectic form ω... it's used for

<u>Thm:</u> (<u>Gromov compactness</u>)

$u_n : \Sigma_n \to X$ sequence of J-holom. curves, $J \in \mathcal{J}(X,\omega)$,
$E(u_n) = \int_{\Sigma_n} u_n^* \omega = \langle [\omega], u_{n*}[\Sigma_n] \rangle$ bounded $\Rightarrow$
$\exists$ subsequence that converges to a <u>stable map</u> $u_\infty : \Sigma_\infty \to X$

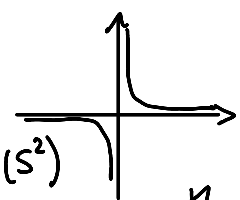
ie: $\Sigma_\infty = \bigcup$ nodal Riemann surfaces



all marked points & nodes are distinct in the domain
(if they come together, create a constant bubble to keep them separated)

<u>Phenomenon:</u> besides possible degeneration of domain (Σ_n, j_n) to a nodal curve, the main phenomenon is <u>bubbling</u> of spheres

Example: $u_n : S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$
$(x_0 : x_1) \longmapsto (x_0 : x_1), (n x_1 : x_0)$

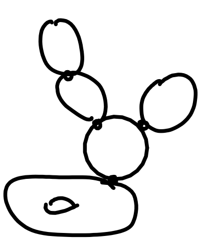


$u_n(S^2)$

(in affine chart $x = x_1/x_0$: $x \longmapsto \left(x, \frac{1}{nx}\right)$ + extend at 0 & ∞)

then away from origin, uniform convergence to $x \longmapsto (x, 0)$
so limit seems to be just 1^{st} coord. axis -- missing part!
but if we reparametrize : $\tilde{x} = nx$, then get $\tilde{x} \longmapsto \left(\frac{\tilde{x}}{n}, \frac{1}{\tilde{x}}\right)$
uniform cv away from ∞ to $\tilde{x} \mapsto \left(0, \frac{1}{\tilde{x}}\right) \to 2^{nd}$ coord axis ✓

<u>Ideas:</u>
- identify bubbling regions = where $\sup |du_n| \to \infty$
- in those regions, rescale domain : $v_n(z) := u_n(z_n^0 + \varepsilon_n z)$,
  $\varepsilon_n \to 0$ suitably chosen $\Rightarrow$ a subsequence of v_n converges to
  a map $v_\infty : \mathbb{C} \to X$, which by removable sing. theorem extends
  to $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$: the bubble !



- intermediate bubbling stages $\Rightarrow$ might need various rescalings to catch all bubbles.
- The process is <u>finite</u> because of energy estimates :
  $E = \int u^* \omega \geq \hbar > 0$ for all nonconstant closed J-hol. curves
  ($\hbar$ ↑ minimum energy)
  and we've assumed an upper bound on total energy.

③

- Assume that we can achieve transversality, even for non-simple curves.

Then: $\overline{\mathcal{M}}_{g,k}(X,J,\beta) = \left\{ \begin{array}{c} \text{(possibly nodal) } J\text{-hol. curves of genus } g \\ \text{representing class } \beta \end{array} \right\} \Big/ \sim$

compact oriented of $\dim_{\mathbb{R}} 2d = 2\langle c_1(TX), \beta\rangle + 2(n-3)(1-g) + 2k$

& carries a fundamental class $[\ ] \in H_{2d}(\overline{\mathcal{M}}_{g,k}(X,J,\beta), \mathbb{Q})$ ⚠ $\mathbb{Q}$-coeffs: due to orbifolding

carries evaluation maps $ev_1, \dots, ev_k : \overline{\mathcal{M}}_{g,k}(X,J,\beta) \longrightarrow X$

$$(\Sigma, j, z_1 \dots z_k, u) \longmapsto u(z_i)$$

→ Gromov-Witten invariants: given $\alpha_1, \dots, \alpha_k \in H^*(X)$, $\sum \deg(\alpha_i) = 2d$,

$$\langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,k}(X,J,\beta)]} ev_1^* \alpha_1 \wedge \dots \wedge ev_k^* \alpha_k$$

or equivalently by Poincaré duality, given cycles $C_1, \dots C_k$ $\quad ([C_i] = PD(\alpha_i))$
and assuming the evaluation maps are transverse to them, this is
$\#(ev_1^{-1}(C_1) \cap \dots \cap ev_k^{-1}(C_k))$ ie. "count of genus g holom. curves in class β passing through $C_1, \dots, C_k$". In general $\in \mathbb{Q}$!

- In the case of a Calabi-Yau 3-fold, dim. formula simplifies to
$\dim \overline{\mathcal{M}}_{g,k}(X,J,\beta) = 2k$ ... ie, holomorphic curves are <u>isolated</u>.

- More about genus 0 GW invariants of Calabi-Yau 3-folds:
how does one build $[\overline{\mathcal{M}}_{0,k}(X,J,\beta)]$? <u>2 flavors of GW theory</u>:

→ <u>symplectic geometry</u>: • <u>Transversality</u>:
  - for simple curves, obtained by choosing J generic
    (seen last time: $\mathcal{J}_{reg}$ dense in $\mathcal{J}(X,\omega)$)
  - multiple covers: always occur with excess dimension $\forall J$
    ($\Sigma' \xrightarrow[\pi]{} \Sigma \to X$, deform covering π .... even though curves in CY 3-folds should be isolated!)

④ Also, multiply covered maps have automorphisms (autom. of covering)

$\Rightarrow \mathcal{M}_{0,k}(X,J,\beta)$ orbifold

$$\begin{array}{c} \Sigma' \xrightarrow{\pi} \Sigma \to X \\ \simeq\downarrow \quad \nearrow_{\pi} \\ \Sigma' \end{array}$$

can restore transv. by using domain-dependent J = function $\mathcal{C} \to \mathcal{J}(X,\omega)$ ← universal curve

$$\begin{array}{c} \mathcal{C} \to \mathcal{J}(X,\omega) \\ \downarrow \\ \bar{\mathcal{M}}_{0,k} \end{array}$$

$u: (\Sigma, j) \to X$, $du + J(u(z), \underline{\underline{z}})\, du\, j = 0$

$z \in \Sigma$

or a perturbation term: $\bar\partial_J u + \nu(z, u(z)) = 0$.

These perturbations "break" the symmetry of the covering π;

they're one reason why # curves $\in \mathbb{Q}$ not $\mathbb{Z}$. (see below)

⚠ case of multiply covered bubbles hardest to deal with; fortunately not an issue for us.

* <u>Compactness</u>: we should stable curves consisting of several components....
  - → in general those should contribute codim. 2 to $\bar{\mathcal{M}}$ if regularity holds (ie. $\mathcal{M}_{0,k}(X,J,\beta)$ defines a <u>pseudocycle</u>, good enough)
  - → in CY 3-fold case they shouldn't contribute at all.

<u>Point</u>: for generic J, we have transversality for simple curves.

Then, given β, $\exists$ <u>finitely many</u> classes of curves with energy $\leq \int_\beta \omega$ (Gromov compactness), and the simple curves in these classes are <u>isolated</u>.

For generic J, evaluation maps are mutually transverse ie. these simple curves are mutually <u>disjoint</u> ... so no bubbled configurations (unless all components cover the same simple curve!)

→ <u>algebraic geometry</u>: prefer to keep J integrable even if it means failure of transversality. [Sometimes transv. <u>does</u> hold...]

For integrable J & fixed j, $D_{\bar\partial}$ is complex linear, honest $\bar\partial$ operator on sections of u^*TX — a holom. bundle! Cokernel $= H^1(\Sigma, u^*TX)$

⑤

Assume u immersion for simplicity: then $u^*TX = T\Sigma \oplus u^*N$ ← normal bundle

tangent part $H^1(\Sigma, T\Sigma)$ is taken care of by letting j vary

(or trivial for $g=0$...). Remaining cokernel: $Obs_u := H^1(\Sigma, u^*N)$.

(if u not immersed, Obs_u = quotient of $H^1(\Sigma, u^*TX)$)

Putting these together over the moduli space of stable maps, get

$$\begin{array}{ll} \underline{Obs} & \text{obstruction sheaf} \\ \downarrow & \\ \overline{\mathcal{M}}_{g,k}(X,J,\beta) & \text{algebraic space (\underline{stack})} \end{array}$$

A perturbation of the holom. curve eqn to $\bar\partial_J u = \nu$ yields a section $\pi_{\mathrm{coker}\,\bar\partial}(\nu)$ of $\underline{Obs}$; the perturbed moduli space is its zero set.

This lets us define a homology class $[\overline{\mathcal{M}}_{g,k}(X,J,\beta)]^{\mathrm{virt}} \in H_{2d}(\overline{\mathcal{M}}_{g,k}(X,J,\beta);\mathbb{Q})$

virt. fundamental class.

E.g. if $\overline{\mathcal{M}}_{g,k}(X,J,\beta)$ smooth but excess dimensional, $\underline{Obs}$ bundle

$\rightsquigarrow [\overline{\mathcal{M}}_{g,k}(X,J,\beta)]^{\mathrm{virt}} = e(\underline{Obs})$ Euler class.