① Lecture 3:

Last time: deformation of (X,J) given by $\{s\in\Omega^{0,1}(X,TX^{1,0}) / \bar\partial s+\frac{1}{2}[s,s]=0\}$

quotient by $\mathrm{Diff}(X) \rightsquigarrow$ <u>1st order</u>: $\mathrm{Def}_1(X,J)=H^1(X,TX)$, but obstructions $\in H^2(X,TX)$

Thm (Bogomolov-Tian-Todorov)

> X compact Calabi-Yau with $H^0(X,TX)=0 \Rightarrow$ deformations of X are unobstructed, ie. a $\mathcal{M}_{cx}(X)$ is locally smooth w/ tangent space $\simeq H^1(X,TX)$.
>
> assuming $\mathrm{Aut}(X,J)=1$

For CY mfolds, $TX\simeq\Omega_X^{n-1}$, $v\mapsto \iota_v\Omega$ so $H^0(X,TX)=H^{n-1,0}\simeq H^{0,1}$ ← assume 0 (see below)

$H^1(X,TX)\simeq H^{n-1,1}$ ← deform^{n}

$H^2(X,TX)\simeq H^{n-1,2}$ ← obstructions)

★ <u>Recall</u>: <u>Hodge theory</u> on compact Kähler manifolds:

- Kähler metric $\rightsquigarrow *$ operator $\rightsquigarrow$ adjoints $d^*=-*d*$, $\bar\partial^*=-*\partial*$

  Laplacians $\Delta=dd^*+d^*d$, $\bar\square=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$: $\underline{\Delta=2\bar\square}$

- Every cohomology class has a unique harmonic representative

  $\bar\partial$-cohomology $\qquad\qquad$ $\bar\square$-harmonic

  (Hodge decomp. theorem)

- so $H^k_{dR}(X,\mathbb{C})\cong \ker(\Delta:\Omega^k(X,\mathbb{C})\circlearrowleft)$

  $=\ker(\bar\square:\Omega^k(X,\mathbb{C})\circlearrowleft)$

  $=\bigoplus_{p+q=k}\ker(\bar\square:\Omega^{p,q}\circlearrowleft)\cong\bigoplus_{p+q=k}H^{p,q}_{\bar\partial}(X)$

- $*$ gives $H^{p,q}\simeq H^{n-q,n-p}$; complex conj: $H^{p,q}\simeq\overline{H^{q,p}}$

  so Hodge diamond
$$\begin{array}{ccccc} & & h^{n,n} & & \\ & & \vdots & & \\ h^{n,0} & & & & h^{0,n} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$
  is symmetric.

- For a CY n-fold, $H^{p,0}\underset{\text{Hodge}*}{\simeq}H^{n,n-p}=H^{n-p}(X,\Omega^n_X)\underset{CY}{\simeq}H^{n-p}(X,\mathcal{O}_X)=H^{0,n-p}$

  so $h^{p,0}=h^{n-p,0}$.

For a CY 3-fold

Under assumption $h^{1,0}=0$, get
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

② • <u>Another interpretation</u> of Kodaira-Spencer map for Calabi-Yaus:

$$\begin{array}{ccc} \mathcal{X} & \supset & X \\ \downarrow & & \downarrow \\ S & \ni & 0 \end{array}$$

family of deformations of $(X,J) \rightsquigarrow (X,J_t)_{t\in S}$

$c_1(K_X)=0$ (deform. inv't) and $H^{0,1}=0$ assumed

$\Rightarrow K_{X_t} \simeq \mathcal{O}_{X_t}$ holomorphically even after deformation (so all (X,J_t) are Calabi-Yau)

Then $\exists [\Omega_t] \in H^{n,0}_{J_t}(X) \subset H^n(X,\mathbb{C})$. Q^n: how does it depend on t?

↑ holom. vol. form

given $\frac{\partial}{\partial t} \in T_0 S$, $\frac{\partial}{\partial t}\Omega_t \in \Omega^{n,0} \oplus \Omega^{n-1,1}$ because of

<u>Thm:</u> (Griffiths transversality) (proved last time)

$$\alpha_t \in \Omega^{p,q}(X,J_t) \Rightarrow \frac{\partial}{\partial t}\alpha_t \in \Omega^{p,q}+\Omega^{p+1,q-1}+\Omega^{p-1,q+1}$$

Now: $\frac{\partial \Omega_t}{\partial t}$ is d-closed (since Ω_t d-closed)

$$\Rightarrow \left(\frac{\partial\Omega_t}{\partial t}\right)^{(n-1,1)} \text{ is } \bar\partial\text{-closed} \Rightarrow \exists \left[\frac{\partial\Omega_t^{(n-1,1)}}{\partial t}\right] \in H^{n-1,1}(X)$$

* For fixed Ω_0, this is indep't of choice of Ω_t. Indeed, could rescale to $f(t)\Omega_t$, but then $\frac{\partial}{\partial t}(f(t)\Omega_t) = \frac{\partial f}{\partial t}\underset{(n,0)}{\Omega_t} + f(t)\frac{\partial\Omega_t}{\partial t}$ ← (n-1,1) part scales linearly

* As seen above, $H^{n-1,1}(X) = H^1(X,\Omega^{n-1}_X) \simeq H^1(X,TX)$

The identification $\underset{v \mapsto i_v\Omega}{TX \simeq \Omega^{n-1}_X}$ also depends on choice of Ω;

the image of $\frac{\partial\Omega_t}{\partial t}$ in $H^1(X,TX)$ is indep't of choices and $\equiv$ Kodaira-Spencer map.

* Hence: for $\theta \in H^1(X,TX)$ deform. of complex structure,

$$\theta.\Omega \in H^1(X,\Omega^n_X\otimes TX) \simeq H^1(X,\Omega^{n-1}_X) = H^{n-1,1}(X)$$

and $[\nabla_\theta \Omega^{(n-1,1)}] \in H^{n-1,1}(X)$ are the same thing

Iterating to 3rd order variation ... on a CY-3 fold,

$$\langle\theta_1,\theta_2,\theta_3\rangle := \int_X \Omega\wedge(\theta_1.\theta_2.\theta_3.\Omega) = \int_X \Omega\wedge\nabla_{\theta_1}\nabla_{\theta_2}\nabla_{\theta_3}\Omega$$

(Need 3 derivatives before we can hit a nontrivial (0,3) component...)

Pseudoholomorphic curves: (reference: McDuff-Salamon book)

(X^{2n},ω) symplectic manifold, J compatible almost-$\mathbb{C}$ structure
($J^2=-1$, $\omega(\cdot,J\cdot)$ Riem. metric)

(Σ,j) Riemann surface of genus g, $z_1,\dots z_k\in\Sigma$ marked points
Moduli space $\mathcal{M}_{g,k} = \{(\Sigma,j,z_1,\dots z_k)\}/\text{biholomorphism}$ $\quad(\dim_{\mathbb{C}}=3g-3+k)$

Main case for us: (S^2,j), $0,1,\infty$: $\mathcal{M}_{0,3}=\{pt\}$ so we won't discuss moduli space further.

Def: $u:\Sigma\to X$ is <u>J-holomorphic</u> if $J\circ du = du\circ j$
i.e. $\bar\partial_J u = \frac{1}{2}(du + J\,du\,j) = 0 \in \Gamma(\Sigma, \Omega^{0,1}_\Sigma\otimes u^*TX)$

Def. $\mathcal{M}_{g,k}(X,J,\beta) = \left\{(\Sigma,j,z_1,\dots z_k),\ u:\Sigma\to X \;/\; \begin{array}{l} u_*[\Sigma]=\beta \\ \bar\partial_J u=0\end{array}\right\}/\sim$
$\beta\in H_2(X)$

(equivalence relation: $\phi:\Sigma\xrightarrow{\sim}\Sigma'$, $\phi(z_i)=z_i'$,
$$\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[d, "\phi"'] & X \\ \Sigma' \arrow[ur, "u'"'] & \end{tikzcd}$$
)

i.e. zero set of a section
$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathrm{Map}(\Sigma,X)_\beta\times\mathcal{M}_{g,k} \arrow[u, bend left, "\bar\partial_J"] \end{tikzcd}$$
$\mathcal{E}$ vector bundle, $\mathcal{E}_u = \Gamma(\Sigma,\Omega^{0,1}_\Sigma\otimes u^*TX)$

More precisely, look at $W^{k+1,p}$ maps, and $\mathcal{E}_u = W^{k,p}(\Sigma,\Omega^{0,1}_\Sigma\otimes u^*TX)$
$\rightsquigarrow$ Banach bundle over a Banach manifold

The linearized operator $D_{\bar\partial}: W^{k+1,p}(\Sigma,u^*TX)\times T\mathcal{M}_{g,k}\to W^{k,p}(\Sigma,\Omega^{0,1}\otimes u^*TX)$
$$D_{\bar\partial}(v,j') = \bar\partial v + \tfrac{1}{2}\nabla_v J\cdot du\circ j + \tfrac{1}{2}J\cdot du\cdot j'$$

is Fredholm, of $\text{index}_{\mathbb{R}} = 2d := 2\langle c_1(TX),\beta\rangle + n(2-2g) + \underbrace{(6g-6+2k)}_{\dim \mathcal{M}_{g,k}}$

Q: transversality? i.e. can we get <u>$D_{\bar\partial}$ to be onto</u> at pts of $\mathcal{M}_{g,k}(X,J,\beta)$?
say u is <u>regular</u> if $D_{\bar\partial}$ onto at u.

(if so then $\mathcal{M}_{g,k}(X,J,\beta)$ is smooth of dimension $2d$)

(4)

Def: $u : \Sigma \to X$ is <u>simple</u> ("somewhere injective") if $\exists\, z \in \Sigma$ s.t. $\begin{cases} du(z) \text{ injective} \\ u^{-1}(u(z)) = \{z\} \end{cases}$

otherwise, u factors through a covering $\Sigma \to \Sigma' \to X$

$\mathcal{M}^*_{g,k}(X,J,\beta) = \{\text{simple } J\text{-hol. curves}\}$

Thm: $\mathcal{J}^{reg}(X,\beta) = \{ J \in \mathcal{J}(X,\omega) \,/\, \text{every simple } J\text{-hol. curve in class } \beta \text{ is regular}\}$
is a Baire subset in $\mathcal{J}(X,\omega)$
For $J \in \mathcal{J}^{reg}(X,\beta)$, $\mathcal{M}^*_{g,k}(X,J,\beta)$ is smooth of real dim. $2d$
and carries a natural orientation.

⚠ in general $\mathcal{M}_{g,k}$ <u>orbifold</u> (Σ with automorphisms)

<u>Proof sketch:</u>

- consider $\bar\partial_J u = 0$ as eqn on $Map(\Sigma, X) \times \mathcal{M}_{g,k} \times \mathcal{J}(X,\omega) \ni (u, j, J)$
then linearization is surjective for all simple maps.
(while fails for multiple covers...)
"univ. moduli space" $\tilde{\mathcal{M}}^* \xrightarrow[\pi_J]{} \mathcal{J}(X,\omega)$: projection to J is Fredholm,
$\Rightarrow$ by Sard-Smale a generic J is a regular value.
and then $\mathcal{M}^*_{g,k}(X,J,\beta)$ is smooth.
- orientation: need orient^n on $\ker(D_{\bar\partial})$. If J is integrable
then $D_{\bar\partial}$ is $\mathbb{C}$-linear and $\exists$ natural orientation. Can still do it in general.

* <u>Moreover</u>: $\forall J_0, J_1 \in \mathcal{J}^{reg}(X,\beta)$, $\exists$ (dense set of choices of) path $\{J_t\}_{t\in[0,1]}$ s.t.
$\coprod_{t\in[0,1]} \mathcal{M}^*_{g,k}(X,J_t,\beta)$ smooth cobordism between $\mathcal{M}^*_{g,k}(X,J_0,\beta)$ and $\mathcal{M}^*_{g,k}(X,J_1,\beta)$
but we need a <u>compactness</u> result, else "# curves" not indep^t of $J \in \mathcal{J}_{reg}$!

* So far we haven't used the symplectic form ω much ... it's used for

<u>Thm:</u> (<u>Gromov compactness</u>)
$u_n : \Sigma_n \to X$ sequence of J-holom. curves, $J \in \mathcal{J}(X,\omega)$,
$E(u_n) = \int_{\Sigma_n} u_n^*\omega = \langle [\omega], u_{n*}[\Sigma_n] \rangle$ bounded $\Rightarrow$
$\exists$ subsequence that converges to a <u>stable map</u> $u_\infty : \Sigma_\infty \to X$