(I) **Recall:** $\mathbb{CP}^1$ is mirror to $\left(\mathbb{C}^*,\ W = z + \frac{e^{-\Lambda}}{z}\right)$ $\quad \Lambda = 2\pi \int_{\mathbb{CP}^1} \omega$

Landau-Ginzburg model = noncompact Kähler mfld + holomorphic function W ("superpotential")

Homological mirror symmetry: (M. Kontsevich '98):

$$\begin{cases} D^{\pi}\mathcal{F}uk(\mathbb{CP}^1) \simeq H^0 MF(W) & \text{matrix factorizations} \\ D^b Coh(\mathbb{CP}^1) \simeq D^b \mathcal{F}uk(\mathbb{C}^*, W). & \text{"Fukaya-Seidel" category} \end{cases}$$

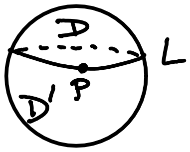
Fukaya category: actually a collection indexed by "charge" $\lambda \in \mathbb{C}$.

$\mathcal{F}uk(\mathbb{CP}^1, \lambda) = \{$ weakly unobstructed Lagrangians with $m_0 = \lambda . [L] \}$

This is an honest A_∞-cat. (m_0's cancel, Floer differential $\partial^2 = 0$) whereas from λ to λ' we'd have $\partial^2 = \lambda' - \lambda$.

- Ex: (L = circle, ∇) is weakly unobstructed, $m_0 = W(L, \nabla). [L]$

However: $HF(L,L) = 0$ unless L = equator & $hol(\nabla) = \pm id$.



$$\partial([p]) = z . ev_{0*}([\mathcal{M}_2(L,[D])] \cap ev_1^{-1}(p)) + z'. \text{ (same with } D')$$
$$= z . [L] - z' . [L].$$

Hence unit $[L] \in \operatorname{im} \partial$ unless $z = \frac{e^{-\Lambda}}{z}$ ie. $z = \pm e^{-\Lambda/2}$ ie. (equator, $\pm$).

For (equator, $\pm$), contrib^{ns} of pairs of symmetric discs cancel exactly and $HF^*(L,L) \simeq H^*(S^1; \mathbb{C})$ as a $\mathbb{Z}/2$-graded vector space

However product structure is deformed: $m_2([p],[p]) = \pm e^{-\Lambda/2} [L]$

ie. multiplicatively $HF^*(L,L) \simeq \mathbb{C}[t]/t^2 = \pm e^{-\Lambda/2}$.

Matrix factorization of $W - \lambda$, $\lambda \in \mathbb{C}$:=

$\mathbb{Z}/2$-graded projective module Q over $R = \mathbb{C}[M]$ (ring of regular functions)

$+\ \delta \in End^1(Q)$ s.t. $\delta^2 = (W - \lambda)\, id_Q$

or equivalently, $Q_0 \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} Q_1$, $\quad \begin{matrix} \delta_0 . \delta_1 = (W-\lambda)\, id_{Q_1} \\ \delta_1 . \delta_0 = (W-\lambda)\, id_{Q_0} \end{matrix}$

$Hom_{MF}(Q,Q')$ = $\mathbb{Z}/2$-graded group of prehom's, eg. $Hom^0 = \left\{ \begin{tikzcd} Q_0 \arrow[r, shift left, "\delta_0"] \arrow[d, "f_0"'] & Q_1 \arrow[l, shift left, "\delta_1"] \arrow[d, "f_1"] \\ Q'_0 \arrow[r, shift left, "\delta'_0"] & Q'_1 \arrow[l, shift left, "\delta'_1"] \end{tikzcd} \right\}$

∂ differential = $f \mapsto \delta' . f \pm f . \delta$ $\qquad$ (Hom^1 =)

Homology category $H^0MF(W-\lambda)$: hom $= H^0(\mathrm{Hom}_{MF}, \partial)$

$=$ "chain maps" up to homotopy

- $H^0MF(W-\lambda)$ is a triangulated cat.

Fact: ‖ $H^0MF(W-\lambda)=0$ (i.e. all M.F.'s are $\simeq 0$) unless $\lambda \in \mathrm{crit}(W)$.

- on the other hand, if looking at hom's from $MF(W-\lambda)$ to $MF(W-\lambda')$, then ∂ doesn't square to 0: $\partial^2(f) = \delta'^2 f - f\delta^2 = (\lambda-\lambda')f$.

 so again we have just a collection of cat's indexed by $\lambda\in\mathbb{C}$.

- Ex: $W = z + \dfrac{e^{-\Lambda}}{z}$ has critical points $\pm e^{-\Lambda/2}$, crit. values $\pm 2e^{-\Lambda/2}$

$$W \pm 2e^{-\Lambda/2} = z \pm 2e^{-\Lambda/2} + \frac{e^{-\Lambda}}{z} = (z\pm e^{-\Lambda/2})\left(1 \pm \frac{e^{-\Lambda/2}}{z}\right)$$

gives a MF $Q_\pm = \left\{ \mathbb{C}[z^{\pm1}] \underset{1\pm e^{-\Lambda/2}z^{-1}}{\overset{z\pm e^{-\Lambda/2}}{\rightleftarrows}} \mathbb{C}[z^{\pm1}] \right\}$

$$\mathrm{End}_{H^0MF}(Q_\pm) \simeq \left\{ \begin{array}{ccc} R & \rightleftarrows & R \\ \downarrow f & & \downarrow f \\ R & \rightleftarrows & R \end{array} \right\} \Big/ \text{homotopy}$$

mult. by some $f\in\mathbb{C}[z^{\pm1}]$

$$\begin{tikzcd} R \arrow[r, shift left] & R \arrow[l, shift left] \arrow[dl, "h"] \\ R \arrow[r, shift left] & R \arrow[l, shift left] \end{tikzcd}$$

gives
$$\begin{tikzcd} R \arrow[r, shift left] \arrow[d, "(z\pm e^{-\Lambda/2})h"'] & R \arrow[l, shift left] \arrow[d, "(z\pm e^{-\Lambda/2})h"] \\ R \arrow[r, shift left] & R \arrow[l, shift left] \end{tikzcd}$$

same for other dir^n

$$\mathrm{Hom}_{H^0MF}(Q_\pm, Q_\pm) \simeq \mathbb{C}[z^{\pm1}] \Big/ \langle z\pm e^{-\Lambda/2}\rangle \simeq \mathbb{C}$$

same for $\mathrm{Hom}_{H^0MF}(Q_\pm, Q_\pm[1]) \simeq \mathbb{C}$.

Multiplication structure: can check matches that of $HF^*(L,L)$.

The other half of mirror symmetry:

- $D^bCoh(\mathbb{CP}^1)$ is <u>generated</u> by $\mathcal{O}(-1)$ and $\mathcal{O}$, ie. smallest full subcat. of $D^bCoh(\mathbb{CP}^1)$ containing $\mathcal{O}(-1)$ and $\mathcal{O}$ & closed wrt shifts & cones is all of $D^bCoh(\mathbb{CP}^1)$

special case of a thm. of <u>Beilinson</u>: ‖ $D^bCoh(\mathbb{CP}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}\rangle$.

③ <u>Idea:</u> diagonal Δ is the transverse zero set of $s = \sum_{i=0}^{n} \frac{\partial}{\partial x_i} \otimes y_i$

section of $T(-1) \boxtimes \mathcal{O}(1) := \pi_1^*(T\mathbb{CP}^n \otimes \mathcal{O}(-1)) \otimes \pi_2^* \mathcal{O}(1)$ on $\underset{[x_0..x_n]}{\mathbb{P}^n} \times \underset{[y_0..y_n]}{\mathbb{P}^n}$

take Koszul resolution $\Rightarrow$ in $\mathbb{P}^1$-case get

$$0 \to \Omega^1(1) \boxtimes \mathcal{O}(-1) \longrightarrow \mathcal{O} \boxtimes \mathcal{O} \longrightarrow \mathcal{O}_\Delta \to 0 \quad \text{in } D^b Coh(\mathbb{P}^1 \times \mathbb{P}^1).$$

Now $\mathcal{E} \in D^b Coh(X \times Y) \rightsquigarrow \Phi^{\mathcal{E}} : D^b Coh(X) \to D^b Coh(Y)$

$$\mathcal{F} \longmapsto R\pi_{2*}(L\pi_1^* \mathcal{F} \overset{L}{\otimes} \mathcal{E})$$

exactness $\Rightarrow \Phi^{\mathcal{O}_\Delta}(\mathcal{F}) \simeq \mathcal{F}$ sits in exact triangle with

$\Phi^{\Omega^1(1) \boxtimes \mathcal{O}(-1)}(\mathcal{F}) \simeq R\Gamma(\Omega^1(1) \otimes \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{O}(-1)$ copies of $\mathcal{O}(-1)$

and $\Phi^{\mathcal{O} \boxtimes \mathcal{O}}(\mathcal{F}) \simeq R\Gamma(\mathcal{F}) \otimes_{\mathbb{C}} \mathcal{O}$ copies of $\mathcal{O}$. $\blacktriangle$

Algebra of exc. collection $\langle \mathcal{O}(-1), \mathcal{O} \rangle$: $\mathcal{A} := End^*(\mathcal{O}(-1) \oplus \mathcal{O}) =$
$$\begin{tikzcd} \underset{\mathcal{O}(-1)}{\bullet} \arrow[loop above, "\mathbb{C}\, id"] \arrow[r, "V"', "\dim = 2"'] & \underset{\mathcal{O}}{\bullet} \arrow[loop above, "\mathbb{C}\, id"] \end{tikzcd}$$

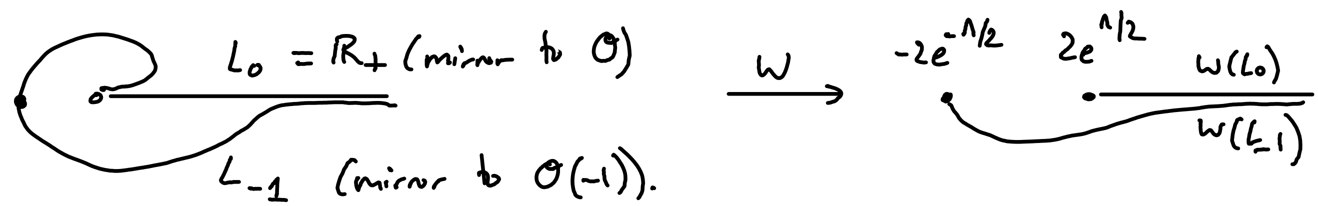
$D^b Coh(\mathbb{CP}^1) \simeq$ der cat. of (f.g.) $\mathcal{A}$-modules

- Fukaya cat. of $(\mathbb{C}^*, W = z + \frac{e^{-\Lambda}}{z})$: objects $=$ <u>admissible</u> Lagrs. L (+ flat ∇)

ie. L possibly noncompact Lagr. submfld, $W_{|L}$ proper, $W_{|L} \in \mathbb{R}_+$ outside a compact subset

[many slightly different definitions ...]

<u>Ex:</u>



<u>perturbation:</u> L admissible $\rightsquigarrow L^{(a)}$ Ham. isotopic, $W \in \mathbb{R}_+ + ia$ at ∞
$a \in \mathbb{R}$

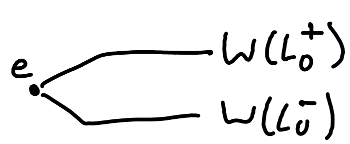
(e.g. in good cases, Ham. flow of $X_{Re W} = \nabla Im W$)

$Hom(L, L') := CF^*(L^{(a)}, L'^{(a')})$ for $a > a'$

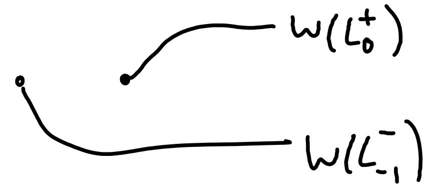
Floer differential ok (compactness holds by max. principle)

$(m_k)_{k \geq 2}$ similarly defined by perturbing Lagrangians in decreasing $Im W$ order

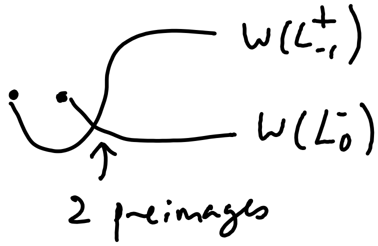
<u>Ex:</u>



$Hom(L_0, L_0) \simeq \mathbb{C} \cdot e$ ($e =$ unit for mult.).



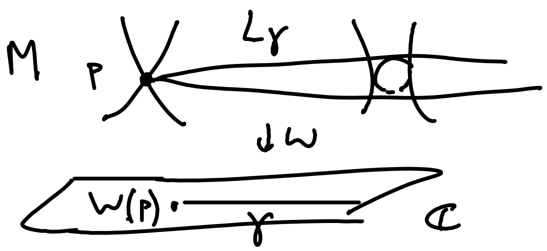
$\mathrm{Hom}(L_0, L_{-1}) = 0$



$\mathrm{Hom}(L_{-1}, L_0) \simeq \mathbb{C}^2$.

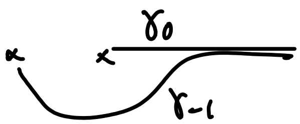
* $\mathcal{F}uk(W)$ is generated by L_{-1} & L_0. This is a special case of

<u>Thm</u> (Seidel) ‖ If crit pts of W are isolated & nondegenerate, (ω exact), then $\mathcal{F}uk(W)$ is generated by a collection of Lefschetz thimbles.



γ arc in $\mathbb{C}$ with end pt a crit val. $W(p)$

$\rightsquigarrow L_\gamma :=$ set of points in $W^{-1}(\gamma)$ s.t. parallel transport along distribution $(T\text{Fiber})^\perp = \mathrm{span}_{\mathbb{C}}(\nabla \mathrm{Re}\, W)$ converges to p.

Here 

gives collection L_{-1}, L_0. with $\mathrm{End}(L_{-1} \oplus L_0) \simeq \mathcal{A}$

$\rightsquigarrow D^b\mathcal{F}uk(W) \simeq D^b(\text{mod-}\mathcal{A}) \simeq D^b\mathrm{Coh}(\mathbb{CP}^1)$. ▲

This strategy of proof of HMS essentially extends to all toric Fano (using more general collections of line bundles / adm. Lagrangians) [~ M. Abouzaid's thesis].

* How did we get these L_{-1}, L_0? SYZ again...

section of $\mathcal{O}(1)$, $s(x_0 : x_1) = x_0$ has norm $|s|^2 = \dfrac{|x_0|^2}{|x_0|^2 + |x_1|^2}$ wrt natural Hermitian metric h, ie. in affine chart, $|s| = (1+|z|^2)^{-1/2}$

$\Rightarrow$ in triv^n given by s, the Chern connection of $(\mathcal{O}(1), h)$ is

$$\nabla = d + \partial \log h(s,s) = d - \partial \log(1+|z|^2) = d - \frac{\bar{z}\, dz}{1+|z|^2} \quad (\text{curv.} = -i\omega_{FS})$$

⑤ Similarly on $\mathcal{O}(k)$, $\nabla = d - k\frac{\bar{z}\,dz}{1+|z|^2}$ (tensor product connection)

Restriction of ∇ to circle $S^1(r)$ is flat with holonomy $\exp(i\theta(r))$

$$i\theta(r) = \int_{S^1(r)} \frac{-k\bar{z}\,dz}{1+|z|^2} = -k\int_0^{2\pi} \frac{ir^2 e^{-i\theta} e^{i\theta}\,d\theta}{1+r^2} = \frac{-2\pi i k r^2}{1+r^2}$$

i.e. $\theta(r)$ goes from 0 to $-2\pi k$ as r from 0 to ∞.

SYZ transform $\rightsquigarrow$ $L_k := \{\phi(r)\, e^{i\theta(r)},\ r \in (0,\infty)\} \subset \mathbb{C}^*$

↑ transform in radial direction

$(\phi(r) = e^{-2\pi\,\mathrm{area}(D^2(r))})$

twists $(-k)$ times around origin [see: K.W. Chan].

⚠ small lie: actual SYZ mirror is $\{e^{-\Lambda} < |z| < 1\}$,
extend to $\mathbb{C}^*$ anyway for simplicity.