① Recall: X, J, ω, Ω (almost) Calabi-Yau

$\rightsquigarrow M = \{(L,\nabla) \,/\, L \subset X$ special Lagr. torus, ∇ flat $U(1)$ conn./gauge$\}$

$$T_{(L,\nabla)} M \cong \left\{ (v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L, i\mathbb{R}) \,/\, -\iota_v \omega + \frac{i\alpha}{2\pi} \in \mathcal{H}^1_\psi(L, \mathbb{C}) \right\}$$

Complex vector space $\Rightarrow$ integrable complex str. $J^\vee$ on M, $z = e^{-2\pi\omega(\beta)} \cdot hol_{\partial\beta}$

- Holom. $(n,0)$-form: $\check{\Omega}((v_1, i\alpha_1) \dots (v_n, i\alpha_n)) = i^{-n} \int_L \left(-\iota_{v_1}\omega + \frac{i\alpha_1}{2\pi}\right) \wedge \dots \wedge \left(-\iota_{v_n}\omega + \frac{i\alpha_n}{2\pi}\right)$
- Kähler form: ($J^\vee$-compatible) $\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) := \frac{1}{2\pi} \int_L \alpha_2 \wedge \iota_{v_1} \operatorname{Im}\Omega - \alpha_1 \wedge \iota_{v_2} \operatorname{Im}\Omega$

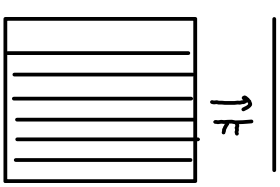
  assuming $\int_L \Omega = 1$.

Now: consider $\pi^\vee : M \to B$, $(L,\nabla) \mapsto L$: the fibers are sLag (easy to check!).

This is meant to be dual to $\pi : X \to B$

⚠ Note however: $\pi^\vee$ has a canonical zero section $\{(L, \nabla = d)\}$ which is S-Lagrangian, and a complex conj. (involution $\nabla \mapsto \nabla^*$)
Construction can't be involutive unless π already had such features.

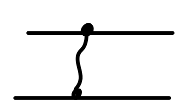
Example: consider $T^2 = \mathbb{C}/\mathbb{Z} + i\rho\mathbb{Z}$, $\Omega = dz$,
$\omega = \frac{\lambda}{\rho} dx \wedge dy$ so $\int_{T^2} \omega = \lambda$



L sLag $\Leftrightarrow \operatorname{Im} dz_{|L} = 0 \Leftrightarrow L$ parallel to real axis.

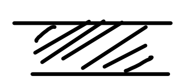
Then: $T^2 \xrightarrow[\pi]{} S^1$ $\quad L_t = \{y = t\}$

- complex affine structure: affine coord $\equiv t$
  $=$ integral of $\operatorname{Im}\Omega$ on arc



  so size of S^1 base is ρ.

- symplectic: affine coord $=$ sympl. area



  so size of S^1 base is λ.

Mirror symm exchanges the two.

②

Coords. on mirror: y and θ = holonomy of connection $\quad y \in \mathbb{R}/\rho\mathbb{Z}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$

Complex structure $J^\vee$: holom. coord is $e^{-2\pi\frac{\lambda}{\rho}y}\, e^{i\theta}$ (see above)

or taking $\frac{1}{2\pi i}\log$, $\quad \check{z} = \frac{\theta}{2\pi} + i\frac{\lambda}{\rho}y \in \mathbb{C}/\mathbb{Z} + i\lambda\mathbb{Z}$

$(\Omega^\vee = d\check{z})$

$\omega^\vee = \frac{1}{2\pi}\, dy \wedge d\theta \qquad$ area $= \rho$

- <u>SYZ transformation:</u>

Lagrangian section of π $\longleftrightarrow$ Connection (hermitian)

$L = \{x = f(y)\}$, $f: \mathbb{R}/\rho\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ $\qquad \check{\nabla} = d + if(y)\,d\theta + ih(y)\,dy$.

with flat conn. $\nabla = d + ih(y)\,dy$, $\quad h: \mathbb{R}/\rho\mathbb{Z} \to \mathbb{R}$

(on some locally trivialized line bundle $\mathcal{L}$)

<u>globally</u>, $\deg(\mathcal{L}) = \deg(f: S^1 \to S^1)$.

Note changing triv^{n} by $\times e^{i\theta}$ changes $\check{\nabla} \leftrightarrow \check{\nabla} + i\,d\theta$ i.e. $f \leftrightarrow f+1$. ✓

Holom. structure $\bar\partial = \check{\nabla}^{0,1}$.

Not visible in T^2, but in higher dimensional tori, Lagr. $\Longleftrightarrow$ holomorphic

$L = \{x = f(y)\}$ Lagrangian, $f: \mathbb{R}^n/\Lambda \to \mathbb{R}^n/\mathbb{Z}^n$ $\longleftrightarrow$ Herm. conn.

$\nabla = d + i\sum_j h_j(y)\,dy_j$ flat, $\quad h: \mathbb{R}^n/\Lambda \to \mathbb{R}^n$ $\qquad \check{\nabla} = d + i\sum_j f_j(y)\,d\theta_j + i\sum_j h_j(y)\,dy_j$

holomorphic iff $F = i\sum_{jk}\frac{\partial f_j}{\partial y_k}\,dy_k \wedge d\theta_j + i\sum_{jk}\frac{\partial h_j}{\partial y_k}\,dy_k \wedge dy_j$

is of type $(1,1)$, ie. $J^\vee$-invariant.

Recall $J^\vee(d\theta_j) = dy_j$ (up to scaling)

$\rightsquigarrow$ $\check{\nabla}$ <u>holomorphic</u> iff
$$\begin{cases} \dfrac{\partial f_j}{\partial y_k} = \dfrac{\partial f_k}{\partial y_j} & \Leftrightarrow \sum f_j(y)\,dy_j \text{ closed } \Leftrightarrow L \text{ \underline{Lagrangian}} \\[2ex] \dfrac{\partial h_j}{\partial y_k} = \dfrac{\partial h_k}{\partial y_j} & \Leftrightarrow \sum h_j(y)\,dy_j \text{ closed } \Leftrightarrow \nabla \text{ \underline{flat}} \end{cases}$$

③ • For more general Calabi-Yaus, finding SLag fibrations and "dualizing" them is much harder... apart from tori, the only fairly easy case is the K3 surface.

K3 surface := simply connected cx. surface with $K_X \simeq \mathcal{O}_X$

eg: degree 4 surface $\{P_4(x_0 \dots x_3) = 0\} \subset \mathbb{CP}^3$

or double cover of $\mathbb{P}^1 \times \mathbb{P}^1$, $\{z^2 = P_{4,4}((x_0:x_1),(y_0:y_1))\} \subset \mathrm{Tot}(\mathcal{O}(2,2))$

Hodge diamond
$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$
(mirror is also a K3)

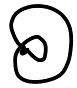
<u>Point</u>: K3 is <u>hyperkähler</u>, ie. can find 3 cx. structures $IJ = -JI = K$
3 kähler forms $\omega_I, \omega_J, \omega_K$
for same metric g.

<u>Idea</u>: find g Ricci-flat kähler metric (exists by Yau's thm), $\Omega \in \Omega^{2,0}$ ($|\Omega| = 1$)

$$\begin{cases} \omega_I = \omega & (1,1) \text{ for } I \\ \omega_J = \mathrm{Re}\,\Omega & (2,0)+(0,2) \text{ for } I \\ \omega_K = \mathrm{Im}\,\Omega & \end{cases}$$

give hyperkähler structure.
($\omega_I, \omega_J, \omega_K$ pointwise orthonormal, self dual, covariantly constant).

• Some (not all) K3 surfaces admit fibration by elliptic curves
(<u>ex</u>: double cover of $\mathbb{P}^1 \times \mathbb{P}^1$: project to $\mathbb{P}^1$ factor).

Typically 24 nodal sing fibers 

If we have such a fibration, with section:

Fibers are I-complex and hence Special Lagrangian both for $(\omega_J, \Omega_J = \omega_K + i\omega_I)$ and $(\omega_K, \Omega_K = \omega_I + i\omega_J)$.

(I-complex curves are calibrated by ω_I
while ω_J, ω_K vanish on $(1,1)_I$ - planes !).

Hence elliptic fibration $\rightsquigarrow$ SLag fibrations on (X, J) & (X, K)

Under suitable assumptions, these are mirror to each other.

(mirror corresp. = HK rotation + Fourier-Mukai functor + HK rotation)

$D^b \mathrm{Coh}\, E \to D^b \mathrm{Coh}\, E^\vee$

$\mathcal{E} \mapsto \pi_{2*}(\pi_1^* \mathcal{E} \otimes \mathcal{P})$