① <u>Recall:</u> (X, ω, J, Ω) Calabi-Yau (strict, or almost CY: $|\Omega|_g = \psi \in C^\infty(L, \mathbb{R}_+)$)

<u>Def:</u> L^n is <u>special Lagrangian</u> if $\omega_{|L} = 0$, $\operatorname{Im} \Omega_{|L} = 0$

(after normalizing Ω so $\int_L \Omega \in \mathbb{R}_+$, else ask $\operatorname{Im}(e^{-i\phi}\Omega)_{|L} = 0$ for some constant phase ϕ).

<u>1st order deformations:</u> $v \in C^\infty(NL)$ normal v.f.

$\rightsquigarrow \beta = -\iota_v \omega \in \Omega^1(L, \mathbb{R})$

$\tilde{\beta} = \iota_v \operatorname{Im} \Omega = \psi \cdot *\beta \in \Omega^{n-1}(L, \mathbb{R})$

Deformation is special Lagr. iff $d\beta = 0$ and $d\tilde{\beta} = 0$, i.e.

1st order deforms $\simeq \mathcal{H}^1_\psi(L, \mathbb{R}) := \{\beta \in \Omega^1(L, \mathbb{R}) / d\beta = 0,\ d^*(\psi\beta) = 0\}$

every class in $H^1(L, \mathbb{R}) \ni$ unique ψ-harm. representative.

<u>Thm:</u> (McLean / Joyce)

Deformations are unobstructed, i.e. moduli space of SLags. is a <u>smooth</u> manifold B with $T_L B \cong \mathcal{H}^1_\psi(L, \mathbb{R})$. ($\simeq H^1(L, \mathbb{R})$)

- We have 2 canonical isoms. $T_L B \xrightarrow{\sim} H^1(L, \mathbb{R})$, $v \mapsto [-\iota_v \omega]$ "symplectic" and $T_L B \xrightarrow{\sim} H^{n-1}(L, \mathbb{R})$, $v \mapsto [\iota_v \operatorname{Im} \Omega]$ "complex"

<u>Def:</u> An <u>affine structure</u> on a mfld N is a set of coord. charts with transition functions in $GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

<u>Corollary:</u> B carries two natural affine structures

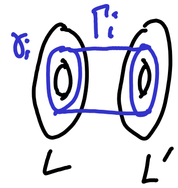
We'll see: "Mirror symmetry = interchange of the affine structures"

- Case of interest to us: special Lagr. <u>tori</u> $\rightsquigarrow$ then $\dim H^1 = n$.

Usual harmonic 1-forms for flat metric on $L = T^n$ have no zeroes (pointwise form basis of $T^*L \simeq NL$); standing assumption: this holds for ψ-harmonic 1-forms wrt $g_{|L}$ too.

Then a nbd of L is fibered by special Lagr. deformations of L, i.e. locally $T^n \to U \subset X$, $\pi: U \to B$ SLag fibration

② ★ Local affine coordinates: pick basis $\gamma_1 \cdots \gamma_n$ of $H_1(L,\mathbb{Z})$



$\leadsto x_i = \int_{\Gamma_i} \omega$ affine coordinates on B for sympl. affine str. (= flux for deform^n of L).

Dually, $\gamma_1^* \cdots \gamma_n^*$ basis of $H_{n-1}(L,\mathbb{Z}) \leadsto$

$$x_i^* = \int_{\Gamma_i^*} \mathrm{Im}\,\Omega \quad \text{affine coords for complex affine structure}$$

This only works locally: globally there's monodromy. The linear part $\in GL(n,\mathbb{Z})$ is given by monodromy of the SLag family: $\pi_1(B,*) \to GL(H^1(L,\mathbb{Z}))$ / $GL(H^{n-1}(L,\mathbb{Z}))$ (Poincaré dual of each other $\Rightarrow$ get transpose monodromies)

★ Prototype construction of mirror pair:

B affine mfld $\leadsto$ lattice $\Lambda \subset TB$ ($\Leftrightarrow$ integer vectors in affine charts)

Then TB/Λ torus bundle / B carries a natural cx. structure (J(base) = fiber ...)

T^*B/Λ^* carries a natural sympl. structure

MS exchanges complex mfld $TB/\Lambda \longleftrightarrow$ sympl. mfld T^*B/Λ^*.

In our case, B carries 2 affine structures with mutually dual monodromies:

$$\begin{array}{ccc} TB & \xrightarrow{\sim} & T^*B \\ \text{cx.}\ \wr\!\wr & & \wr\!\wr\ \text{sympl.} \\ H^{n-1}(L,\mathbb{R}) & \underset{\text{P.D.}}{\simeq} & H_1(L,\mathbb{R}) \\ \cup & & \cup \end{array}$$

$$\Lambda_c = H^{n-1}(L,\mathbb{Z}) \simeq H_1(L,\mathbb{Z}) = \Lambda_s^*$$

i.e. $\underset{\text{cx. geom.}}{TB/\Lambda_c} \simeq \underset{\text{sympl. geom.}}{T^*B/\Lambda_s^*}$

and dually for the mirror geometry.

★ Let's construct the candidate mirror more explicitly: [see also Hitchin]

Let $M = \{(L,\nabla) \,/\, L \text{ special Lagr.}, \nabla \text{ flat } U(1) \text{ conn.}/\text{gauge}\}$

(ie. $\nabla = d + A$, $A \in \Omega^1(L, i\mathbb{R})$, $dA = 0$, mod exact forms)

$$T_{(L,\nabla)} M \cong \{(v, i\alpha) \in C^\infty(NL) \oplus \Omega^1(L; i\mathbb{R}) \,/\, -\iota_v \omega \in \mathcal{H}^1_\psi(L,\mathbb{R}),\ d\alpha = 0\} / 0 \times \mathrm{Im}\, d$$

③ $T_{(L,\nabla)}M \cong \{(v,i\alpha)\in C^\infty(NL)\oplus\Omega^1(L,i\mathbb{R}) \,/\, -\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L,\mathbb{C})\}$

complex vector space ⇒ M carries a natural almost-ℂ structure $J^\vee$.

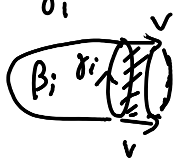
Prop: $J^\vee$ is integrable.

Pf: enough to give local holom. coordinates.

$\gamma_1\dots\gamma_n$ basis of $H_1(L,\mathbb{Z})$; assume each $\gamma_i=\partial\beta_i$, $\beta_i\in H_2(X,L)$

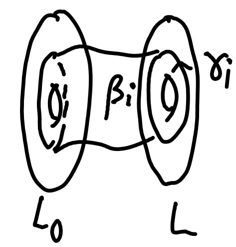
Then set $z_i(L,\nabla) := \exp\left(-\int_{\beta_i}\omega\right)\,\mathrm{hol}_\nabla(\gamma_i)\in\mathbb{C}^*$

$\rightsquigarrow d\log z_i(v,i\alpha) = -\int_{\gamma_i}\iota_v\omega + i\int_{\gamma_i}\alpha_i = \langle[\underbrace{-\iota_v\omega+i\alpha_i}_{H^1(L,\mathbb{C})}],[\gamma_i]\rangle$



basis of $T^*M^{1,0}$ ✓

If such β_i don't exist, do the same with



⚠ EVERYTHING UP TO FACTORS OF 2π

- Holom. (n,0)-form: $\check\Omega((v_1,i\alpha_1)\dots(v_n,i\alpha_n)) = \int_L(-\iota_{v_1}\omega+i\alpha_1)\wedge\dots\wedge(-\iota_{v_n}\omega+i\alpha_n)$

(if take γ_i "standard" basis above, then in above coords. $\check\Omega=\prod d\log z_i$)

- Kähler form: $\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) := \int_L \alpha_2\wedge\iota_{v_1}\mathrm{Im}\,\Omega - \alpha_1\wedge\iota_{v_2}\mathrm{Im}\,\Omega$

[recall we've normalized $\int_L\Omega=1$]

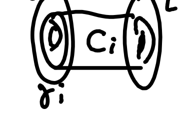
Prop: $\omega^\vee$ is a Kähler form compatible with $J^\vee$

Pf: pick $[\gamma_i]$ basis of $H_{n-1}(L,\mathbb{Z})$, $[e_i]$ basis of H_1 s.t. $e_i\cdot\gamma_j=\delta_{ij}$.

Then $\forall\, a\in H^1(L)$, $b\in H^{n-1}(L)$, $\langle a\cup b,[L]\rangle = \sum_i\langle a,e_i\rangle\langle b,\gamma_i\rangle$ (*)

(think: e_i = i^{th} coord. axis, γ_i = i^{th} hyperplane)

let $p_i=\int_{C_i}\mathrm{Im}\,\Omega$,



(affine coords for ℂ affine structure)

$\theta_i=\int_{e_i}A$ (i.e. $\mathrm{hol}_{e_i}(\nabla)=e^{i\theta_i}$)

④

Then $dp_i : (v,\alpha) \mapsto \int_{\gamma_i} \iota_v \operatorname{Im}\Omega = \langle [\iota_v \operatorname{Im}\Omega], \gamma_i \rangle$

$d\theta_i : (v,\alpha) \longmapsto \int_{e_i} \alpha = \langle [\alpha], e_i \rangle$

and $(*) \Rightarrow \omega^\vee = \sum dp_i \wedge d\theta_i$ ($\Rightarrow$ closed).

Now: $\omega^\vee\big((v_1,\alpha_1),(v_2,\alpha_2)\big) = \int_L \psi\left(\langle \alpha_1, \iota_{v_2}\omega \rangle_g - \langle \alpha_2, \iota_{v_1}\omega \rangle_g \right)$

$\Rightarrow \omega^\vee\big((v_1,\alpha_1), J^\vee(v_2,\alpha_2)\big) = \int_L \psi\left(\langle \alpha_1, \alpha_2 \rangle_g + \langle \iota_{v_1}\omega, \iota_{v_2}\omega \rangle_g \right)$

clearly a Riemannian metric ✓