① Recall: <u>Motivation for SYZ conjecture:</u>

<u>Q:</u> ‖ how does one build a mirror $X^\vee$ of a given Calabi-Yau manifold X?

<u>Observe:</u> HMS says $D^bCoh(X^\vee) \simeq D^\pi Fuk(X)$

$p \in X^\vee$ point $\leftrightarrow \mathcal{O}_p \in D^bCoh(X^\vee) \leftrightarrow \mathcal{L}_p \in D^\pi Fuk(X)$.

$X^\vee$ = moduli space of skyscraper sheaves in $D^bCoh(X^\vee)$
= moduli space of certain objects in $D^\pi Fuk(X)$.

$HF^*(\mathcal{L}_p, \mathcal{L}_p) \cong Ext^*(\mathcal{O}_p, \mathcal{O}_p) \simeq H^*(T^n; \mathbb{C}) \Rightarrow$ reasonable guess:

$\rightsquigarrow$ ‖ <u>generic</u> points of $X^\vee$ correspond to isomorphism classes of (L, ∇), LCX Lagr. torus ∇ $U(1)$-flat conn.

(some points of $X^\vee$ might still only correspond to objects of the derived Fukaya category).

⚠ The Strominger-Yau-Zaslow conj. (1996) builds on this and gives a richer geometric picture (get <u>both</u> cx. & sympl. geometry on each of $X, X^\vee$) by picking a preferred representative of the isom. class of (L,∇) (doesn't always exist ⚠).

<u>SYZ conj:</u> ‖ $X, X^\vee$ carry dual fibrations by <u>special Lagrangian tori</u>

Ie:
$$\begin{array}{ccc} T^n \to X & , & \check{T}^n \to \check{X} \\ \downarrow \pi & & \downarrow \pi^\vee \\ B & & B \end{array}$$
where $\check{T} = Hom(\pi_1 T, U(1))$ dual torus

ie. $\check{X} = \{(L,\nabla) \,/\, L$ fiber of π, $\nabla \in hom(\pi_1 L, U(1))\}$ & vice-versa.

<u>Special Lagrangian</u> := $\omega_{|L} = 0$ <u>and</u> $Im(\Omega)_{|L} = 0$
holom. volume form

We'll look more into it but there are several warnings:

* Constructing SLag torus fibrations is difficult & usually impossible.
(Joyce, Haase-Zharkov, Gross-Siebert, ...)

② <u>general slogan</u>: A LCSL degeneration should give rise to a SLag fibration (the CY metric collapses to B). Still very hard.

(also note: different choice of LCSL degeneration should give a different SLag fibration and hence a different mirror).

* SLag fibration will usually have singularities ⟹ dual fibration not well-defined. A related issue = "instanton corrections"

So conjecture as stated mostly applies to tori... needs to be adjusted in general.

<u>Special Lagrangian submanifolds</u>:

X, ω, J Kähler, g Kähler metric, $\Omega \in \Omega^{n,0}$ holom. volume form

strict Calabi-Yau: g Ricci-flat, $|\Omega|_g = \text{const.}$ vs. almost-CY: $|\Omega|_g = \psi \in C^\infty(X, \mathbb{R}_+)$

$\Omega \wedge \bar{\Omega} = c(n)\, \omega^n$ $\qquad$ $\Omega \wedge \bar{\Omega} = \psi^2 c(n)\, \omega^n$.

(<u>point</u>: curvature of Chern connection on $\Omega^{n,0}$ ≡ Ricci form; strict CY ⟺ $\nabla\Omega = 0$)

<u>Fact</u>: ‖ $L \subset X$ Lagrangian submfld ⟹ $\Omega_{|L} \in \Omega^n(L, \mathbb{C})$ is of the form $\Omega_{|L} = e^{i\varphi}\psi\, \mathrm{vol}_{g|L}$ with $e^{i\varphi}: L \to S^1$ <u>phase function</u>

(<u>Pf</u>: linear algebra! at a point $p \in L$, $\exists$ basis of T_pX s.t. $(T_pX, \omega_p, J_p, T_pL) \simeq (\mathbb{C}^n, \omega_0, J_0, \mathbb{R}^n)$, and $\Omega_p = e^{i\varphi(p)}\psi(p)\, dz_1 \wedge \dots \wedge dz_n$)

<u>Def</u>: ‖ L is <u>special Lagrangian</u> if the phase function is constant.

Then $\int_L \Omega \in e^{i\varphi}\mathbb{R}_+$. Given $[L] \in H_n(X, \mathbb{Z})$, normalize Ω so that $\int_{[L]} \Omega = 1$.

⤳ <u>Def'</u>: ‖ L is special Lagrangian iff $\operatorname{Im} \Omega_{|L} = 0$.

(and then $\operatorname{Re} \Omega_{|L} = \psi \cdot \mathrm{vol}_L$, up to suitable choice of orient$^{\underline{n}}$ of L)

<u>Rmk 1</u>: in strict CY case, special Lagrangians are <u>calibrated</u> & hence volume-minimizing in their homology class: $\operatorname{Re} \Omega_{|\pi} \le \mathrm{vol}_{g|\pi}$ $\forall \pi$ n-plane, with equality iff π special Lagrangian. Hence

$[\operatorname{Re}\Omega] \cdot [L] = \int_L \operatorname{Re}\Omega \le \int_L \mathrm{vol}_g = \mathrm{vol}(L)$ with equality iff S-Lag.

(3) <u>Rmk 2:</u> $c_1(TX) = 0 \Rightarrow \exists$ global $\mathbb{Z}$-cover of Lagr. grassmannian of X.

Can describe a graded Lagr. plane as:

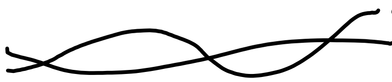
$$\begin{cases} \Pi \subset TX \text{ Lagr. plane} \\ \varphi \in \mathbb{R} \text{ real lift of phase } \arg(\Omega_{|\Pi}) \end{cases}$$

For a general Lagr. $L \subset X$, $e^{i\varphi}: L \to S^1$ may not lift to $\varphi: L \to \mathbb{R}$. Obstruction = homotopy class in $[L, S^1] = H^1(L, \mathbb{Z})$.

Up to factor of 2 this is exactly the <u>Maslov class</u> μ_L.

For L special Lagr., $\mu_L = 0$ automatically ($\Rightarrow$ graded lifts exist CF^* are $\mathbb{Z}$-graded)

<u>Deformation of special Lagrangians:</u>



$L_t = \exp(tv)$, $v \in C^\infty(NL)$ normal vector field

L

deformation of L

Q^n: when is L_t special Lagrangian? $\varphi_t = \exp(tv): L \to X$

$L_t = \varphi_t(L)$.

- <u>Lagrangians</u> need $\omega_{|L_t} = 0 \ \forall t$, ie. $\varphi_t^* \omega = 0$

1^{st} order condition: $\frac{d}{dt}(\varphi_t^* \omega)_{|t=0} = L_v \omega = d(\iota_v \omega)$

$\beta = -\iota_v \omega \in \Omega^1(L, \mathbb{R})$ should be closed $\quad d\beta = 0$

- <u>special:</u> need $\operatorname{Im} \Omega_{|L_t} = 0$ ie. $\varphi_t^*(\operatorname{Im}\Omega) = 0$

1^{st} order: $\frac{d}{dt}_{|t=0}(\varphi_t^* \operatorname{Im}\Omega) = L_v \operatorname{Im}\Omega = d(\iota_v \operatorname{Im}\Omega)$

$\tilde\beta = \iota_v \operatorname{Im}\Omega \in \Omega^{n-1}(L, \mathbb{R})$ should also be closed. $d\tilde\beta = 0$

$\rightarrow$ Relation between $\beta, \tilde\beta$? go back to pointwise linear algebra:

$T_pX \simeq \mathbb{C}^n$, J_0, ω_0, $T_pL = \mathbb{R}^n$, $\Omega_{|p} = \psi\, dz_1 \wedge \dots \wedge dz_n$

$v = \sum a_i \frac{\partial}{\partial y_i} \rightarrow \beta = \sum a_i\, dx_i$

$\tilde\beta = \sum a_i \cdot (-1)^{i-1} \psi\, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$

Hence $\tilde\beta = \psi * \beta$. (Hodge $*$ for $g_{|L}$)

In strict CY case, $\tilde\beta = *\beta$, so $d\beta = d\tilde\beta = 0 \Leftrightarrow \beta$ harmonic.

④

Prop: ‖ 1^{st} order deformations of a special Lagr. submanifold in a strict CY $\cong \mathcal{H}^1(L,\mathbb{R})$.

In almost-CY, 1^{st} order deforms $\simeq \mathcal{H}^1_\psi(L,\mathbb{R}) := \{\beta\in\Omega^1(L,\mathbb{R}) / d\beta=0,\ d^*(\psi\beta)=0\}$

still true that every class in $H^1(L,\mathbb{R}) \ni$ unique ψ-harm. representative.

(Idea: redo Hodge decomp. theorem but with $\Omega^1 \xrightarrow{(d,\psi^{-1}d^*\psi)} \Omega^2\oplus\Omega^0$
$= (d,d^*)$ + order 0

or... if dim. $n\neq 2$, ψ-harmonic for $g \Leftrightarrow$ harmonic for $\psi^{\frac{2}{n-2}}g$)

Thm: (McLean / Joyce)

‖ Deformations are unobstructed, ie. moduli space of SLags. is a <u>smooth</u> manifold B with $T_LB \cong \mathcal{H}^1_\psi(L,\mathbb{R})$. $(\simeq H^1(L,\mathbb{R}))$.

Pf: locally near L, deforms $\overset{\exp}{\Longleftrightarrow}$ normal vector fields. Consider the Banach bundle $\mathcal{E}$ over $\mathcal{U}\subset W^{k,p}(L,NL)$ with fiber at v $W^{k-1,p}(L,\Lambda^2T^*L)\oplus W^{k-1,p}(L,\Lambda^nT^*L)$, and the section $s(v) = (\exp(v)^*\omega,\ \exp(v)^*\operatorname{Im}\Omega)$; Then $B = s^{-1}(0)$.

ω, $\operatorname{Im}\Omega$ closed $\Rightarrow s(v)$ always takes values in closed forms, and looking at Lie derivatives, since $s(0)=0$, <u>exact</u> forms.

$\mathcal{F}\subset\mathcal{E}$ Banach subbundle of exact forms, then s is a Fredholm section of $\mathcal{F}$, and $ds(0)\circ(\omega^\#)^{-1} : \beta \mapsto (-d\beta,\ d(\psi *\beta))$ is onto
$\begin{pmatrix}\omega^\# : NL \xrightarrow{\sim} T^*L \\ v\mapsto -i_v\omega\end{pmatrix}$ $\Rightarrow s^{-1}(0)$ smooth. ◭