(1) Last time: HMS for $T^2$ (area $\lambda$) / elliptic curve ($\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, $\tau = i\lambda$)

Lagrangians of slope $(p,q)$ + flat $U(1)$-connections $\longleftrightarrow$ Vector bundles of rank $p$, degree $-q$. (or: for $(p,q)=(0,-1)$, skyscraper sheaves)

We compared $m_2$'s using theta functions.

* To actually prove HMS, need to understand (& match) leftover part of $A_\infty$-structure on derived category: <u>Massey products</u>.

Look at a special case: in a tri-cat. $\mathcal{D}$, consider objects & morphisms $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} X_4$ where $g\cdot f = 0$, $h \cdot g = 0$,

& assume also $\hom(X_1, X_3[-1]) = \hom(X_2, X_4[-1]) = 0$

We still have an element $m_3(h,g,f) \in \hom(X_1, X_4[-1])$:

let $K$ be s.t.
$$\begin{tikzcd} K \arrow[r] & X_2 \arrow[dl, "g"] \\ X_3 \arrow[u, "{[1]}"] & \end{tikzcd}$$
distinguished (ie. $K[1] = \mathrm{Cone}(g)$)

then $g\cdot f = 0 \Rightarrow f$ factors through $X_1 \xrightarrow{\bar{f}} K \to X_2$

$h \cdot g = 0 \Rightarrow h$ factors through $X_3 \to K[1] \xrightarrow{\bar{h}} X_4$

[argument: $\hom(X_1,K) \to \hom(X_1,X_2) \xrightarrow{g} \hom(X_1,X_3)$ exact $\Rightarrow f$ factors

also $\hom(X_1,X_3[-1]) = 0$ $\Rightarrow$ factors uniquely ].

Now $m_3(h,g,f) := m_2(\bar{h}[-1], \bar{f})$: $X_1 \xrightarrow{\bar{f}} K \xrightarrow{\bar{h}[-1]} X_4[-1]$

Why is that related to $m_3$ from $A_\infty$ structure?

Lift $f,g,h$ to "chain level" $A_\infty$-tri. cat. of (twisted) complexes,

then can take $K = \{X_2 \xrightarrow{g} X_3[-1]\}$ and now

$\bar{f}$, $\bar{h}[-1]$ are
$$\begin{tikzcd} X_1 \arrow[d, "f"'] & \\ X_2 \arrow[r, "g"] & X_3[-1] \arrow[d, "{h[-1]}"] \\ & X_4[-1] \end{tikzcd}$$

$m_2^{Tw}(\bar{h}[-1], \bar{f}) = m_3(h,g,f)$

by def$^{n}$ of $m_2^{Tw}$

(insert $\delta$'s everywhere).

②

( lengthier but strictly equivalent derivation:

can take $K = \{\underset{\deg 0}{X_2} \xrightarrow{g} \underset{\deg 1}{X_3[-1]}\}$, let $e =$
$$\begin{tikzcd} X_2 \arrow[r, "g"] \arrow[d, "id"] & X_3[-1] \\ X_2 \arrow[r, "g"'] & X_3[-1] \end{tikzcd}$$

Then $m_1(e) =$
$$\begin{tikzcd} X_2 \arrow[r, "g"] \arrow[dr, "g"'] & X_3[-1] \\ X_2 \arrow[r, "g"'] & X_3[-1] \end{tikzcd}$$
$\Rightarrow$
$$\begin{tikzcd} X_1 \arrow[d, "f"'] & \\ X_2 \arrow[r, "g"] \arrow[dr, "g"'] & X_3 \\ X_2 \arrow[r, "g"'] & X_3 \arrow[d, "h"] \\ & X_4 \end{tikzcd}$$
$f, h$ are $\underline{m_1\text{-closed}}$ & $g = m_1(e)$

$m_3(h, g, f) = m_3(h, m_1(e), f)$
$= m_2(h, m_2(e, f))$ + (other terms which all vanish)
$= m_2(\bar{h}[-1], \bar{f})$. )

* $\underline{\text{Look at:}}$ $\mathcal{L}$ nontrivial degree 0 line bundle $\downarrow X^\vee$ $\quad$ $p, q \in X^\vee$ distinct generic

$$\mathcal{O} \xrightarrow[f]{\text{1-dim}^l} \mathcal{O}_p \xrightarrow[g]{\text{1-dim}^l} \mathcal{L}[1] \xrightarrow[h]{\text{1-dim}^l} \mathcal{O}_q[1]$$

$\hom(\mathcal{O}_p, \mathcal{L}[1]) = \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{L}) \underset{\text{Serre}}{\simeq} \mathrm{Hom}(\mathcal{L}, \mathcal{O}_p)^\vee \simeq$ fiber of $\mathcal{L}$ at $p$

$\hom(\mathcal{O}, \mathcal{L}[1]) = \mathrm{Ext}^1(\mathcal{O}, \mathcal{L}) = H^1(\mathcal{L}) = 0$ by Riemann-Roch

$\hom(\mathcal{O}_p, \mathcal{O}_q[1]) = 0$

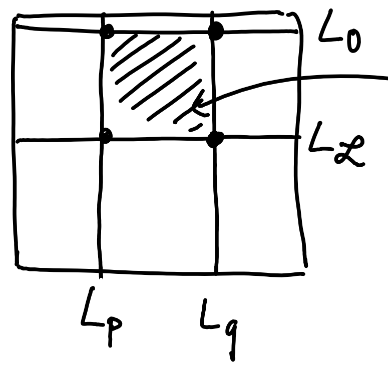
Massey product of generators: $K \simeq \underbrace{\mathcal{L} \otimes \mathcal{O}(p)}_{\text{another deg 1 line bundle}}$ $\quad$ deg 1 line bundle w/ section vanishing at $p$

( $0 \to \mathcal{L} \xrightarrow[s_p]{} \mathcal{L} \otimes \mathcal{O}(p) \to \mathcal{O}_p \to 0$ + rotate exact triangle )
has the extension class $g$ $\quad$ $\Rightarrow$
$$\begin{tikzcd} K \arrow[r] & \mathcal{O}_p \arrow[dl, "g"] \\ \mathcal{L}[1] \arrow[u, "{[1]}"] & \end{tikzcd}$$

$\underline{\text{Hence:}}$ $\bar{f}$ = nontrivial section of deg 1 bundle $K \simeq \mathcal{L} \otimes \mathcal{O}(p)$
$\bar{h}[-1]$ = nontrivial hom from $K$ to $\mathcal{O}_q$ (or rather $K[1] \to \mathcal{O}_q[1]$)
(as long as $\mathcal{L} \otimes \mathcal{O}(p) \not\simeq \mathcal{O}(q)$).

③ $\Rightarrow$ this Massey product is nontrivial, and can be computed and compared with the Fukaya cat. $m_3$:

$L_0 \to L_p \to L_{\mathcal{L}}[1] \to L_q[1]$.




this rectangle is one in a $\mathbb{Z}^2$-family of rectangles that contribute to $m_3$

[see Polishchuk].

With more work one can prove HMS for $T^2$ in this way ...

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<u>Motivation for SYZ conjecture:</u>

<u>Q:</u> how does one build a mirror $X^\vee$ of a given Calabi-Yau manifold $X$?

<u>Observe:</u> HMS says $D^b Coh(X^\vee) \simeq D^\pi Fuk(X)$

In particular, $p \in X^\vee$ point $\leftrightarrow \mathcal{O}_p \in D^b Coh(X^\vee)$

$\leftrightarrow \mathcal{L}_p \in D^\pi Fuk(X)$.

$X^\vee$ = moduli space of skyscraper sheaves in $D^b Coh(X^\vee)$

= moduli space of certain objects in $D^\pi Fuk(X)$.

* What kind of objects?

Recall [Lec. 16]: $Ext^k(\mathcal{O}_p, \mathcal{O}_p) \simeq \Lambda^k V$ ($V \simeq$ tangent space at $p$)

ie. as graded vector space, $Ext^*(\mathcal{O}_p, \mathcal{O}_p) \simeq H^*(T^n; \mathbb{C})$

Recall [Lec. 12]: in good cases $HF^*(L,L) \simeq H^*(L)$

(though in general, if $L$ bounds holom. discs, only related by a spectral sequence)

⚠ should be with $\Lambda$-coefficients, but in good cases can work over a smaller coefficient ring. Since complex side is over $\mathbb{C}$, let's try to use $\mathbb{C}$ as well (set $T = e^{-2\pi}$) and hope for convergence. [Otherwise ... in general recall mirror symm. only holds near LCSL, should have started with a formal family, ie. a scheme over $\Lambda^{\mathbb{C}}$].

- So if we're optimistic & hope $\mathcal{L}_p$ is actually an honest Lagrangian, then it should be a Lagrangian torus.

  In fact there's not enough of these: given $T^n \simeq L \subset X$,

  $U(L) \simeq T^*L$ and Lagr. deformation of $L \simeq$ graphs of closed 1-forms

  Hamiltonian isotopies $\simeq$ graphs of exact 1-forms

  $\Rightarrow$ tangent space to "moduli sp. of Lagrangian tori" (⚠) at $L$ is $\cong H^1(L,\mathbb{R})$.

  For $T^n$ this is real $n$-dim$^{l}$, half what we want.

- However: recall twisted Floer homology for $(L,\nabla)$ [Lec. 14]

  $\nabla =$ flat $U(1)$ conn. on $\underline{\mathbb{C}} \to L$

  ($= d+A$, $A \in \Omega^1(L; i\mathbb{R})$ closed) (mod gauge = exact)

  $\nabla$ affects Floer theory by inserting holonomy factors in disc weights.

$\rightsquigarrow$ actually a more realistic hope is that generic points of $X^\vee$ correspond to isomorphism classes of $(L,\nabla)$, $L \subset X$ Lagr. torus, $\nabla$ $U(1)$-flat conn.

(some points of $X^\vee$ might still only correspond to objects of the derived Fukaya category).

⚠ The Strominger-Yau-Zaslow conj. (1996) builds on this and gives a richer geometric picture (get both cx. & sympl. geometry on each of $X$, $X^\vee$) by picking a preferred representative of the isom. class of $(L,\nabla)$ (doesn't always exist ⚠).