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Lecture 2 — Deformations of complex structures; Hodge theory

Reference: Gross-Huybrechts-Joyce, "CY mflds & related geometries", ch. 14

- (X,J) almost complex $(J^2=-1)$ $\rightsquigarrow$ $TX\otimes\mathbb{C} = TX^{1,0}\oplus TX^{0,1}$

$$v^{1,0}=\tfrac12(v-iJv),\quad v^{0,1}=\tfrac12(v+iJv)$$

similarly, $T^*X\otimes\mathbb{C}\simeq \underset{\text{span}(dz_i)}{T^*X^{1,0}}\oplus \underset{\text{span}(d\bar z_i)}{T^*X^{0,1}}$

$$\Lambda^k T^*X = \bigoplus_{p+q=k}\Lambda^{p,q}T^*X = \underset{\text{notation}}{\Omega^{p,q}(X)}$$

$(TX,J)\simeq TX^{1,0}$ complex vector bundle

- integrability of complex structure $\Leftrightarrow [T^{1,0},T^{1,0}]\subseteq T^{1,0}$

$\Leftrightarrow d=\partial+\bar\partial$ maps $\Omega^{p,q}\to\Omega^{p+1,q}\oplus\Omega^{p,q+1}$

$\Leftrightarrow \bar\partial^2=0$

then TX and assocd bundles are holom. vector bundles

Dolbeault cohomology: E holom. vect bundle $\Rightarrow$

$$C^\infty(X,E)\xrightarrow{\bar\partial}\Omega^{0,1}(X,E)\xrightarrow{\bar\partial}\Omega^{0,2}(X,E)\to\cdots$$

$$\rightsquigarrow H^q_{\bar\partial}(X,E)=\frac{\ker\bar\partial}{\operatorname{Im}\bar\partial}$$

- Deforming J to a nearby J':

$\Omega^{1,0}_{J'}\subseteq T^*X\otimes\mathbb{C}=\Omega^{1,0}_J\oplus\Omega^{0,1}_J$ is the graph of a linear map

$(-s):\Omega^{1,0}_J\to\Omega^{0,1}_J$. Conversely, recover J' from s: if s small enough

then $\Omega^{1,0}_{J'}:=\text{graph}(-s)$, $\Omega^{0,1}_{J'}=\overline{\Omega^{1,0}_{J'}}$ satisfy $T^*X\otimes\mathbb{C}=\Omega^{1,0}_{J'}\oplus\Omega^{0,1}_{J'}$

& set $J'=\begin{pmatrix} i & \\ & -i\end{pmatrix}$

Can also view s as section of $(\Omega^{1,0}_J)^*\otimes\Omega^{0,1}_J\simeq T^{1,0}_J\otimes\Omega^{0,1}_J$

ie. (0,1)-form with values in $T^{1,0}X$

$z_1,\dots,z_n$ local holom. coordinates for $(X,J)\Rightarrow s=\sum s_{ij}\frac{\partial}{\partial z_i}\otimes d\bar z_j$

then basis of (1,0)-forms for J': $dz_i - s(dz_i) = dz_i-\sum_j s_{ij}\,d\bar z_j$

(0,1)-vector fields $\frac{\partial}{\partial\bar z_k}+s\left(\frac{\partial}{\partial\bar z_k}\right)=\frac{\partial}{\partial\bar z_k}+\sum_\ell s_{\ell k}\frac{\partial}{\partial z_\ell}$ pair trivially

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- Integrability?

$\left(\bigoplus_q \Omega_X^{0,q} \otimes TX^{1,0}, \bar\partial\right)$ Dolbeault complex for $TX^{1,0}$ on (X,J)
($\bar\partial$ acts on forms only)

carries a Lie bracket $[\alpha\otimes v, \alpha'\otimes v'] := (\alpha\wedge\alpha')\otimes[v,v']$

$\rightsquigarrow$ diff'l graded Lie algebra (dgLa).

Prop: J' is integrable $\iff$ $\bar\partial s + \frac{1}{2}[s,s] = 0$

Pf: want: $\left[\frac{\partial}{\partial \bar z_i} + \sum_\ell s_{\ell i}\frac{\partial}{\partial z_\ell}, \frac{\partial}{\partial \bar z_j} + \sum_\ell s_{\ell j}\frac{\partial}{\partial z_\ell}\right] \in TX^{0,1}_{J'}$?

$$= \sum_\ell \left(\frac{\partial s_{\ell j}}{\partial \bar z_i}\frac{\partial}{\partial z_\ell} - \frac{\partial s_{\ell i}}{\partial \bar z_j}\frac{\partial}{\partial z_\ell}\right) + \sum_{k,\ell}\left(s_{ki}\frac{\partial s_{\ell j}}{\partial z_k}\frac{\partial}{\partial z_\ell} - s_{kj}\frac{\partial s_{\ell i}}{\partial z_k}\frac{\partial}{\partial z_\ell}\right)$$

$\in \operatorname{span}\left(\frac{\partial}{\partial z_\ell}\right)\ldots$

$\Rightarrow$ should be zero: want: $\forall i,j,\ell$,

$$\underbrace{\frac{\partial s_{\ell j}}{\partial \bar z_i} - \frac{\partial s_{\ell i}}{\partial \bar z_j}}_{\substack{\text{coeff}^{\text{t}}\text{ of } (d\bar z_i\wedge d\bar z_j)\otimes\frac{\partial}{\partial z_\ell} \\ \text{in } \bar\partial s}} + \underbrace{\sum_k\left(s_{ki}\frac{\partial s_{\ell j}}{\partial z_k} - s_{kj}\frac{\partial s_{\ell i}}{\partial z_k}\right)}_{\substack{\frac{1}{2}\cdot\text{coeff of } d\bar z_i\wedge d\bar z_j\otimes\frac{\partial}{\partial z_\ell} \\ \text{in } [s,s]}} = 0$$

$\blacktriangle$

- We'd like to understand $\mathcal{M}_{cx}(X) = \{J \text{ integrable cx. str. on } X\}/\mathrm{Diff}(X)$
or rather its germ near X

(or: assuming $\mathrm{Aut}(X,J)$ is discrete, near J $\exists$ universal family
$\mathcal{X} \xrightarrow[\pi]{} \mathcal{U} \subset \mathcal{M}_{cx}$, $\mathcal{X}, \mathcal{U}$ complex manifolds, π holomorphic,
fibers of π are $\simeq X$
any other family near J is induced by a classifying map & pullback from $\mathcal{X}$.)

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- $\{\text{integrable } J\text{'s}\} \underset{\text{locally}}{\simeq} \{ s \in \Omega^{0,1}(X, TX^{1,0}) \,/\, \bar\partial s + \frac{1}{2}[s,s] = 0 \}$

but need to quotient by Diff(X): $J \sim \phi^* J$

If ϕ is close to Id, can be written in local coords. as

$$\phi : (z_1, \dots, z_n) \longmapsto (z_1 + f_1(z,\bar z), \dots, z_n + f_n(z,\bar z))$$

then $\phi^* dz_i = dz_i + \sum_j \left(\frac{\partial f_i}{\partial z_j} dz_j + \frac{\partial f_i}{\partial \bar z_j} d\bar z_j \right)$ $\rightsquigarrow$ write s for $\phi^* J$...
$s = -(Id + \partial f)^{-1} \bar\partial f$

or: $\partial\phi : TX^{1,0} \to \phi^* TX^{1,0}$ parts of $d\phi$ that commute / anticommute w/ J
$\bar\partial\phi : TX^{0,1} \to \phi^* TX^{1,0}$

$$\Rightarrow \phi^* dz_i = \underbrace{dz_i \circ \partial\phi}_{(1,0)} + \underbrace{dz_i \circ \bar\partial\phi}_{(0,1)} = \underbrace{(dz_i \circ \partial\phi)}_{(1,0)} \circ (Id + (\partial\phi)^{-1}\bar\partial\phi)$$

i.e. $s = -(\partial\phi)^{-1}\bar\partial\phi$

- Tangent space – infinitesimal deformations ("over Spec $\mathbb{C}[t]/t^2$")

$J(t)$, $J(0) = J \rightsquigarrow s(t) \in \Omega^{0,1}(X, TX^{1,0})$, $\bar\partial s(t) + \frac{1}{2}[s(t), s(t)] = 0$

$\rightsquigarrow s_1 = \frac{ds}{dt}_{|t=0}$ satisfies $\bar\partial s_1 = 0$

Infinitesimal action of diffeomorphisms:

(ϕ_t), $\phi_0 = Id$, $\frac{d\phi}{dt}_{|t=0} = v$ vector field $\rightsquigarrow$

$$\frac{d}{dt}_{|t=0} \left(-(\partial\phi_t)^{-1} \bar\partial\phi_t \right) = -\frac{d}{dt}_{|t=0} (\bar\partial\phi_t) = -\bar\partial v$$

So: first order deformations

$$\mathrm{Def}_1(X,J) = \frac{\ker(\bar\partial : \Omega^{0,1}(X, TX^{1,0}) \to \Omega^{0,2})}{\mathrm{Im}(\bar\partial : C^\infty(X, TX^{1,0}) \to \Omega^{0,1})} = H^1(X, TX^{1,0})$$

In particular, given a family $\begin{matrix} \mathcal{X} \supset X \\ \downarrow \quad \downarrow \\ S \ni 0 \end{matrix}$ of deformations of (X, J) param^d by S

get a map $T_0 S \to H^1(X, TX)$ by looking at 1st order variations of J....

<u>Kodaira-Spencer map</u>

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- Another way to think about this: (X,J) complex mfld $= \left(\bigsqcup U_i\right)/\phi_{ij}$

U_i complex charts, $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ biholomorphisms, $\phi_{ji} = \phi_{ij}^{-1}$, $\phi_{ij}\phi_{jk} = \phi_{ik}$

Then, deforming $(X,J) \iff$ deform gluing maps ϕ_{ij} among holom. maps

to 1^{st} order, this is given by holom. vector fields v_{ij} on $U_i \cap U_j$

& should satisfy $v_{ji} = -v_{ij}$, $v_{ij} + v_{jk} = v_{ik}$ on $U_i \cap U_j \cap U_k$

$\Rightarrow$ Čech 1-cocycle with values in sheaf of holom. tangent vector fields

Mod out by: $\psi_i : U_i \xrightarrow{\sim} U_i$ diffeo, change $\phi_{ij} \rightsquigarrow \psi_j \phi_{ij} \psi_i^{-1}$

to 1^{st} order, v_i holom. vector fields on U_i, affect gluings by

$v_{ij} = v_i - v_j$ ie. Čech coboundary

$\rightsquigarrow$ get again $H^1(X, TX)$

- Obstruction: given a first-order deform^n s_1, can we find an actual deform^n $s(t) = s_1 t + O(t^2)$ (or a formal deform^n $\sum_{n\geq 1} s_n t^n$) ?

Working order by order to solve $\bar\partial s(t) + \frac{1}{2}[s(t), s(t)] = 0$:

$$\bar\partial s_1 = 0$$
$$\bar\partial s_2 + \tfrac{1}{2}[s_1, s_1] = 0$$
$$\bar\partial s_3 + [s_1, s_2] = 0$$
$$\cdots\cdots$$

$\Rightarrow$ need: $[s_1, s_1] \in \operatorname{Im} \bar\partial \subseteq \Omega^{0,2}(X, TX^{1,0})$?

know: $[s_1, s_1] \in \ker \bar\partial$ (since $\bar\partial s_1 = 0$).

$\rightsquigarrow$ primary obstruction: class of $[s_1, s_1]$ in $H^2(X, TX^{1,0})$.

If vanishes then $\exists\, s_2$ s.t. $\bar\partial s_2 + \frac{1}{2}[s_1, s_1] = 0$.

Next obstruction: class of $[s_1, s_2]$ in $H^2(X, TX^{1,0})$

If vanishes then $\exists\, s_3$ ... and so on.

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If it happens that $H^2(X,TX)=0$ then deformations are unobstructed ie. given s_1 lifts to all orders! ($\to \exists$ actual deformation).
For Calabi-Yaus, in general $H^2(X,TX)\neq 0$, but remarkably:

Thm (Bogomolov-Tian-Todorov)

> X compact Calabi-Yau with $H^0(X,TX)=0 \Rightarrow$ deformations of X are unobstructed, ie. $\mathcal{M}_{cx}(X)$ is locally smooth w/ tangent space $\simeq H^1(X,TX)$.
> assuming $Aut(X,J)=1$

(for CY mfolds, $TX\simeq \Omega^{n-1}_X$ so
$v\mapsto \iota_v\Omega$

$H^0(X,TX)=H^{n-1,0}\overset{\text{we'll see}}{\simeq} H^{0,1}$ ← assume 0
$H^1(X,TX)\simeq H^{n-1,1}$ ← defor$^{\text{n}}$
$H^2(X,TX)\simeq H^{n-1,2}$ ← obstructions)

* For Calabi-Yaus, we'll reinterpret Kodaira-Spencer map in terms of $[\Omega]\in H^{n,0}\subset H^n(X,\mathbb{C})$. For this we'll need:

Thm: (Griffiths transversality)

> $\alpha_t\in\Omega^{p,q}(X,J_t)\Rightarrow \frac{\partial}{\partial t}\alpha_t\in\Omega^{p,q}+\Omega^{p+1,q-1}+\Omega^{p-1,q+1}$

Pf: (X,J_t) locally given by $s(t)\in\Omega^{0,1}(X,TX^{1,0})$, $s(0)=0$

In local coords, $T^*X^{1,0}_{J_t}=\text{span}\{dz_i^{(t)}:=dz_i-\sum_j s_{ij}(t)\,d\bar z_j\}$ (seen above)

$$\alpha_t=\sum_{|I|=p,|J|=q}\alpha_{IJ}(t)\,dz_{i_1}^{(t)}\wedge\dots\wedge dz_{i_p}^{(t)}\wedge d\bar z_{j_1}^{(t)}\wedge\dots\wedge d\bar z_{j_q}^{(t)}$$

Take $\frac{\partial}{\partial t}\big|_{t=0}$ & apply product rule: since $s_{ij}(0)=0$, only terms not $\in\Omega^{p,q}$ are $\alpha_{IJ}(0)\,dz_{i_1}\wedge\dots\wedge\left(\sum_j \frac{\partial s_{i_k j}}{\partial t}d\bar z_j\right)\wedge\dots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\dots\wedge d\bar z_{j_q}\in\Omega^{p-1,q+1}$
and similarly (differentiating $d\bar z_{j_k}$) terms in $\Omega^{p+1,q-1}$ ▲