① **Homological mirror symmetry conjecture** (Kontsevich 1994)

$X, X^\vee$ mirror Calabi-Yaus $\Rightarrow$ $D^\pi \mathrm{Fuk}(X) \simeq D^b\mathrm{Coh}(X^\vee)$ and vice versa.

Let's see how this works <u>at homology level</u> in example of T^2 (after Polishchuk - Zaslow)

Consider on symplectic side $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\omega = \lambda\, dx \wedge dy$ (so $\int_{T^2}\omega = \lambda$)

on complex side $X^\vee = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, $\tau = i\lambda$.

- Lagrangian submflds in X can be isotoped (Hamiltonianly) to straight lines with rational slope; likewise, we can arrange flat conn. 1-form to be constant.

We'll see: family of Lagrangians in homology class (p,q) (+ $U(1)$ conn.)
$\longleftrightarrow$ family of rank p vector bundles on $X^\vee$ with $c_1 = -q$.

- $L \to X^\vee$ line bundle $\rightsquigarrow$ pullback to univ. cover $\mathbb{C} \to X^\vee$ is holomorphically trivial; in fact, pullback to $\mathbb{Z}$-cover $\mathbb{C}/\mathbb{Z}$ too.

$$L \simeq \mathbb{C}\times\mathbb{C} \Big/ \begin{array}{l}(z,v)\sim(z+1,v)\\ (z,v)\sim(z+\tau,\varphi(z)v)\end{array}$$ where φ holomorphic, $\varphi(z+1)=\varphi(z)$

<u>Ex.</u> $\varphi(z) = e^{-2\pi i z}e^{-\pi i\tau}$ determines degree 1 line bundle $\mathcal{L}$ with section the theta function $\theta(\tau,z) = \sum_{m\in\mathbb{Z}} e^{2\pi i\left(\frac{\tau m^2}{2}+mz\right)}$

generalization: $$\theta[c',c''](\tau,z) := \sum_{m\in\mathbb{Z}} \exp 2\pi i\left(\tau\frac{(m+c')^2}{2} + (m+c')(z+c'')\right)$$

satisfies $$\theta[c',c''](\tau,z+1) = e^{2\pi i c'}\,\theta[c',c''](\tau,z)$$
$$\theta[c',c''](\tau,z+\tau) = e^{-\pi i\tau}e^{-2\pi i(z+c'')}\,\theta[c',c''](\tau,z)$$

$$\left[\tau\frac{(m+c')^2}{2} + \tau(m+c') + (m+c')(z+c'') = \tau\frac{(m+1+c')^2}{2} - \frac{\tau}{2} + (m+1+c')(z+c'') - (z+c'')\right]$$

②

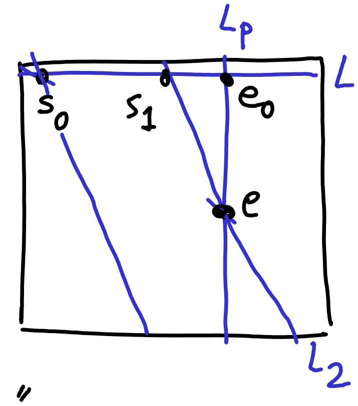
* sections of $\mathcal{L}^{\otimes n}$ are $\theta\left[\frac{k}{n},0\right](n\tau, nz)$. $k=0,\dots,n-1$
* other line bundles are obtained by pullback by translation $z \mapsto z+c''$
  i.e. $\theta[0,c'']$ etc...
* higher rank vector bundles = similar story with matrices – or pushforward line bundle via finite cover $\left(E_x = \bigoplus_{y\in\pi^{-1}(x)} L_y\right)$ .....

<u>A first check:</u> $L_0 = \{(x,0)\}$, $\nabla_0 = d$
"mirror to $\mathcal{O}$"

$L_n = \{(x,-nx)\}$, $\nabla_n = d$
"mirror to $\mathcal{L}^{\otimes n}$"

$L_p = \{(a,y)\}$, $\nabla_p = d + 2\pi i b\, dy$
"mirror to $\mathcal{O}_z$, $z = b + a\tau$"

<u>e.g.</u>

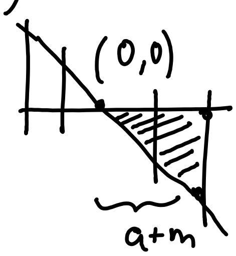


grade (L_0,∇_0) so $\arg(dz_{|TL_0}) = 0$
(L_n,∇_n) $\in(-\frac{\pi}{2},0)$
(L_p,∇_p) $= -\pi/2$

Then $s_k = (\frac{k}{n},0) \in CF^0(L_0,L_n)$
$e = (a,-na) \in CF^0(L_n,L_p)$
$e_0 = (a,0) \in CF^0(L_0,L_p)$
are all in <u>degree 0</u>.

<u>Compute:</u> $m_2(e_n, s_0) = \underline{\ ?\ }\, e_0$?
↖ need to count holomorphic discs in T^2...

observe:
* discs lift to universal cover $\mathbb{R}^2 = \mathbb{C}$
* Maslov index calculation $\Rightarrow$ rigid holom. discs are immersed
  (as maps $D^2 \to \mathbb{C}$, derivative has no zeroes)
* get an ∞ sequence of holom. triangles T_m, $m\in\mathbb{Z}$
  in univ. cover vertices are at
  $(0,0)$, $(a+m, -n(a+m))$, $(a+m,0)$.
  $\Rightarrow$ area $\int_{T_m} \omega = \lambda n (a+m)^2/2$



* holonomy / boundary is $\exp\left(2\pi i \int_{-n(a+m)}^{0} b\, dy\right) = \exp\left(2\pi i n (a+m) b\right)$
* the immersed triangles T_m are all regular (calc. $\bar\partial$-operator...)
* sign calc^{n} : orient^{n} is $+1$ for all T_m (if trivial spin structures)

③

$$\Rightarrow m_2(e,s_0) = \underbrace{\left(\sum_{m\in\mathbb{Z}} T^{\frac{\lambda n}{2}(a+m)^2} e^{2\pi i n(a+m)b}\right)}_{\text{set } T = e^{-2\pi},\ \text{ie. } T^{\lambda} = e^{2\pi i \tau}} e_0$$

$$\Rightarrow \text{coeff}^t = \sum_{m\in\mathbb{Z}} \exp 2\pi i \left(\frac{n\tau m^2}{2} + n(\tau a+b)m + \left(\frac{n\tau a^2}{2} + nab\right)\right)$$

$$= \underbrace{e^{\pi i n\tau a^2} e^{2\pi i nab}}_{\text{change of triv}^{n} \text{ (holomorphic vs. unitary) at } z = \tau a+b} \theta(n\tau, n(\tau a+b))$$

ie: this is $\mathcal{O} \xrightarrow{s_0} \mathcal{L}^n \xrightarrow{ev_z} \mathcal{O}_z$

ev_z : evaluation at z in <u>suitable trivialization</u> - not the one we thought!

To check if this is valid, try same with s_k (intersection at $(\frac{k}{n}, 0)$):

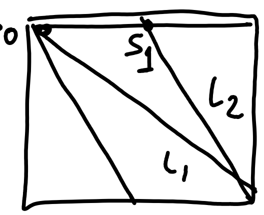
coeff^t of e_0 in $m_2(e, s_k)$ is similarly

$$\sum_{m\in\mathbb{Z}} \exp 2\pi i \left(\frac{n\tau}{2}\left(a+m-\frac{k}{n}\right)^2 + n\left(a+m-\frac{k}{n}\right)b\right)$$

$$= \sum_{m\in\mathbb{Z}} \exp 2\pi i \left(\frac{n\tau}{2}\left(m-\frac{k}{n}\right)^2 + n(\tau a+b)\left(m-\frac{k}{n}\right) + \frac{n\tau a^2}{2} + nab\right)$$

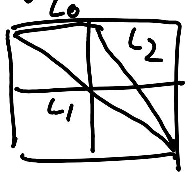
$$= e^{\pi i n\tau a^2} e^{2\pi i nab}\, \theta\left[-\frac{k}{n}, 0\right](n\tau, n(\tau a+b)) \quad \text{ie. ratios match } \frac{s_k(z)}{s_0(z)} \checkmark$$

· Similarly, look at multiplication of sections: simplest example:



$$m_2(\underset{\hom(L_1,L_2)}{s_0}, \underset{\in \hom(L_0,L_1)}{s_0}) = c_0 \underset{\hom(L_0,L_2)}{s_0} + c_1 s_1 \ ?$$

c_0 counts triangles with all vertices at s_0, there's a constant one then



area $= \lambda$, and others ...

$$\rightsquigarrow c_0 = \sum_{n\in\mathbb{Z}} T^{n^2\lambda} = \sum_{n\in\mathbb{Z}} e^{2\pi i\tau n^2}$$

similarly $c_1 = \sum_{n\in\mathbb{Z}} e^{2\pi i\tau(n+\frac{1}{2})^2}$

Corresponds to: $\theta(\tau,z)\,\theta(\tau,z) = \underbrace{\theta[0,0](2\tau,0)}_{c_0}\, \underbrace{\theta[0,0](2\tau,2z)}_{s_0} + \underbrace{\theta[\tfrac{1}{2},0](2\tau,0)}_{c_1}\, \underbrace{\theta[\tfrac{1}{2},0](2\tau,2z)}_{s_1}$

(4) Can do more systematic calculations for more general line bundles and also higher rank bundles $\rightsquigarrow$ build a functor between homology categories & check m_2 is preserved. [Polishchuk-Zaslow]