1. Last time we constructed the der. cat. of an abelian category, e.g. $D^b\mathrm{Coh}(X)$

* <u>Derived Fukaya category</u>: similar story except $\mathcal{F}(M,\omega)$ is an A_∞-category, not an abelian category... construction proceeds via "twisted complexes"

1) additive enlargement $\Sigma\mathcal{A}$: allow formal direct sums and shifts of objects of $\mathcal{A}$

objects $:= \bigoplus_{\text{finite}} X_i[k_i]$ $\quad X_i \in \mathcal{A}$; $k_i \in \mathbb{Z}$; $\quad \hom(\bigoplus_i X_i[k_i], \bigoplus_j Y_j[l_j]) = \bigoplus_{i,j} \hom(X_i, Y_j)[l_j - k_i]$

(write as matrices)

$$m_k(a_k, \dots, a_1)^{ij} = \sum_{i_1 \cdots i_{k-1}} m_k(a_k^{i_{k-1} j}, \dots, a_1^{i i_1})$$

2) twisted complexes: $\mathrm{Tw}\,\mathcal{A}$: objects = twisted complexes (X, δ_X)

$X = \bigoplus_i X_i[k_i] \in \mathrm{ob}\,\Sigma\mathcal{A}$, $\quad \delta_X = (\delta_X^{ij}) \in \hom^1_{\Sigma\mathcal{A}}(X,X)$ degree 1 endo.

s.t.
- δ_X strictly lower-triangular
- $\sum_{k=1}^{\infty} m_k(\delta_X, \dots, \delta_X) = 0$. $\quad$ [generalizes $\delta^2 = 0$].

actually finite by strict triangular

<u>Ex:</u>
$$\begin{tikzcd} X_1[2] \arrow[r, "f"] \arrow[rr, bend right, "h"'] & X_2[1] \arrow[r, "g"] & X_3 \end{tikzcd}$$

$g \in \hom^0(X_2, X_3) = \hom^1(X_2[1], X_3)$
$f \in \hom^0(X_1, X_2)$
$h \in \hom^{-1}(X_1, X_3)$

write $m_1 = \partial$, $m_2 = \cdot$: eqn becomes
$\partial f = 0$
$\partial g = 0$
$g \cdot f = \partial h$.

morphisms of twisted complexes: $\hom_{\mathrm{Tw}\mathcal{A}}((X,\delta_X),(Y,\delta_Y)) = \hom_{\Sigma\mathcal{A}}(X,Y)$

but product structures deformed by differentials:

$$m_k^{\mathrm{Tw}\mathcal{A}}(a_k, \dots, a_1) = \sum_{i_0 \cdots i_k} m^{\Sigma\mathcal{A}}_{k+i_0+\dots+i_k}(\underbrace{\delta_{X_k} \cdots \delta_{X_k}}_{i_k}, a_k, \underbrace{\delta_{X_{k-1}} \cdots \delta_{X_{k-1}}}_{i_{k-1}}, \dots, a_1, \underbrace{\delta_{X_0} \cdots \delta_{X_0}}_{i_0})$$

<u>Ex:</u>
- $a \in \hom(X,Y) \Rightarrow m_1^{\mathrm{Tw}\mathcal{A}}(a) = m_1(a) + m_2(\delta_Y, a) + m_2(a, \delta_X)$ + higher order terms

Think of this as generalizing diff! on prehom of complexes (closed $\leftrightarrow$ "chain map", exact $\longleftrightarrow$ "nullhomotopic")

- $m_2^{\mathrm{Tw}\mathcal{A}}(a_2, a_1) = m_2(a_2, a_1)$ + higher order terms

② $Tw\,\mathcal{A}$ is a triangulated A_∞-category ($\exists$ mapping cones, like usual complexes)

3) cohomology category $D(\mathcal{A}) := H^0(Tw\,\mathcal{A})$ (honest tri. cat.): same objects, but

$$hom(X,Y) := H^0\left(hom^{Tw\,\mathcal{A}}(X,Y),\ m_1^{Tw\,\mathcal{A}}\right) \quad (\text{NB: } hom(X,Y[k]) = H^k(\dots))$$

[analogue of: chain maps up to homotopy]

composition = induced by $m_2^{Tw\,\mathcal{A}}$ on cohomology.

Rmk: there's no localization step wrt quasi-isoms: quasi-isomorphisms are built into the A_∞-structure and already invertible up to homotopy.

- Variant: split-closed der. cat.

$X \in \mathcal{A}$ linear cat., $p \in Hom_{\mathcal{A}}(X,X)$ is idempotent if $p^2 = p$.

Image of p := Y + maps $X \underset{v}{\overset{u}{\rightleftarrows}} Y$ s.t. $uv = id_Y$, $vu = p$

doesn't always exist in $\mathcal{A}$ $\Rightarrow$ need enlargement to achieve this.

Split-closure of $\mathcal{A}$: objects = (X,p), p idempotent endom. of X

$$hom((X,p),(Y,p')) = p'\,hom(X,Y)\,p$$

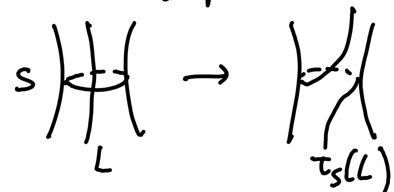
In A_∞ setting, use a more sophisticated approach (Yoneda embedding to A_∞-modules, modules which are quasi-isom. to abstract image of an idempotent).

Geometrically:

- some exact triangles in derived Fukaya category can be understood as Lagr. connected sum / Dehn twist [Seidel, see also FOOO]

Ex: S Lagrangian sphere $\rightsquigarrow$ τ_S Dehn twist $\in Symp(M,\omega)$

easiest in 1-dim'l case:



L Lagr. $\rightsquigarrow$ $\tau_S(L)$ Lagrangian

(in higher dim, defined using geodesic flow in nbd of $S \simeq T^*S$)

Seidel: $\exists$ exact triangle in $DFuk(M)$:

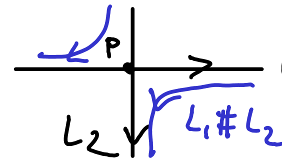
$$\begin{tikzcd} HF^*(S,L)\otimes S \arrow[rr, "t"] & & L \arrow[dl] \\ & \tau_S(L) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

($\rightsquigarrow$ long exact sequence for $HF(L',-)$)

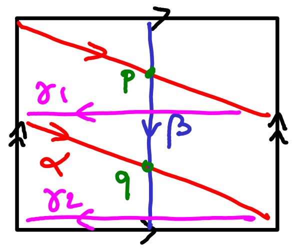
③ Similarly, L_1, L_2 graded Lagrangian, $L_1 \cap L_2 = p$ of index 0

$\rightsquigarrow$ $L_1 \#_p L_2 \simeq \mathrm{Cone}(L_1 \xrightarrow{p} L_2)$

vs. "$L_1[1] \cup_p L_2 \simeq \mathrm{Cone}(L_1 \xrightarrow{0} L_2)$"



So e.g. consider T^2:

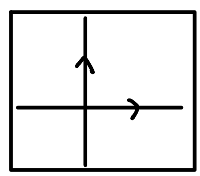


$\mathrm{Cone}(\alpha \xrightarrow{p+q} \beta)$

$\simeq$ disconnected Lagrangian $\gamma_1 \oplus \gamma_2$

If we only started with α & β, der. cat. would have $\gamma_1 \oplus \gamma_2$ but not γ_1 & γ_2 separately; split-closure addresses this.

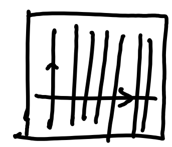
▲ If we start with 2 generators



successive Dehn twists give all homotopy classes of loops on T^2;

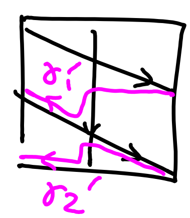
but each homotopy class $\ni$ ∞ly many non-Ham.-isotopic Lagrangians.

To generate $DFuk(T^2)$ as triangulated envelope we need e.g. 1 horiz. loop + ∞ly many vertical loops



On the other hand, α & β as above split generate.

Key point: $\mathrm{Cone}(\alpha \xrightarrow[p+T^q q]{} \beta)$ gives



direct sums of loops that vary continuously within homotopy class

▲ But many cones & idempotents don't have an obvious geom. interpretation.

e.g. Clifford torus $T = \{|x| = |y| = |z|\} \subset \mathbb{CP}^2$ has idempotents $\in HF(T,T)$ without any obvious geometric interpretation.