① **Recall:** equivalence relation on complexes:

Def: $C_\bullet \xrightarrow{f} D_\bullet$ chain map (i.e.
$$\begin{tikzcd} \cdots C_i \arrow[r, "\partial"] \arrow[d, "f"] & C_{i+1} \arrow[r, "\partial"] \arrow[d, "f"] & C_{i+2} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots D_i \arrow[r, "\partial"] & D_{i+1} \arrow[r, "\partial"] & D_{i+2} \arrow[r] & \cdots \end{tikzcd}$$
)
is a <u>quasiisomorphism</u> if the induced maps on cohomology are isomorphisms

This is stronger than $H^*(C_\bullet) \simeq H^*(D_\bullet)$

<u>Ex:</u> $\mathbb{C}[x,y]^2 \xrightarrow[(x,y)]{} \mathbb{C}[x,y]$ and $\mathbb{C}[x,y] \xrightarrow[0]{} \mathbb{C}$

not quasiisomorphic as complexes of $\mathbb{C}[x,y]$-modules even though same H^*

<u>Ex:</u> $\{\mathcal{L}^{-1} \xrightarrow[s]{} \mathcal{O}_X\}$ and $\mathcal{O}_D$ are quasiisomorphic, q-isom = kernel map (similarly with other resolutions of coherent sheaves).

<u>Def^{ns}</u>: • an <u>additive category</u> := • $Hom(A,B)$ abelian groups
• Composition is distributive (bilinear)
• $\exists$ direct sums of objects $A \oplus B$
• $\exists$ zero object 0 $(hom(0,A) = hom(A,0) = 0)$

• <u>abelian category</u> = additive cat. s.t. all morphisms have ker & Coker

[everything defined by univ. properties, e.g. kernel of $A \xrightarrow{f} B$ is $K \to A$ st. $g: C \to A$ factors (uniquely) through K iff $f \circ g = 0$.
In actual examples, ker/coker are always "usual" ones].

$\rightsquigarrow$ in an abelian cat. we have notions of - exact sequence
- cohomology of a complex.

Def: $\mathcal{A}$ abelian category $\rightsquigarrow$ the <u>bounded derived cat.</u> $D^b(\mathcal{A})$:
* objects = bounded (ie., finite length) chain complexes in $\mathcal{A}$
* morphisms = chain maps up to homotopy, localizing wrt quasi-isoms.

• <u>homotopy</u>:
$$\begin{tikzcd} \cdots A_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] & A_i \arrow[r, "d_i"] \arrow[dl, "h_i"'] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & A_{i+1} \arrow[r, "d_{i+1}"] \arrow[dl, "h_{i+1}"'] \arrow[d, shift right, "f_{i+1}"'] \arrow[d, shift left, "g_{i+1}"] & {} \\ \cdots B_{i-1} \arrow[r, "d'_{i-1}"'] & B_i \arrow[r, "d'_i"'] & B_{i+1} \arrow[r, "d'_{i+1}"'] & {} \end{tikzcd}$$

f, g are homotopic $(f \sim g)$ if $\exists h: A \to B[-1]$ st. $f - g = d_B h + h d_A$.
Then look at chain maps$/\sim$

②

- <u>Equivalently</u>: bounded complexes form a differential graded category

morphisms = "prehoms of complexes" $\underline{\mathrm{Hom}}^k(A_\bullet, B_\bullet) = \bigoplus_i \mathrm{Hom}_{\mathcal{A}}(A_i, B_{i+k})$

differential = $f \in \underline{\mathrm{Hom}}^k(A_\bullet, B_\bullet) \Rightarrow \delta(f) = d_B f + (-1)^{k+1} f\, d_A$.

Then chain maps = $\ker(\delta : \underline{\mathrm{Hom}}^0 \to \underline{\mathrm{Hom}}^1)$

nullhomotopic = $\mathrm{Im}(\delta : \underline{\mathrm{Hom}}^{-1} \to \underline{\mathrm{Hom}}^0)$

$\Rightarrow$ we want to consider $H^0\,\underline{\mathrm{Hom}}(A, B)$.

- <u>Localization</u> wrt quasi isoms := formally invert quasi-isos, ie. add extra morphisms s^{-1} wherever s is a quasi-iso.

In other terms, $\mathrm{Hom}_{D^b(\mathcal{A})}(A_\bullet, B_\bullet) = \{ A \underset{\text{quasi iso}}{\xleftarrow{s}} A' \underset{\text{chain map}}{\xrightarrow{f}} B \} / \sim$

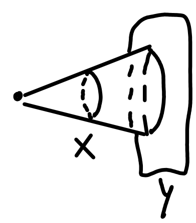
[NB: can skip quotienting by homotopies, because homotopy equivalences are quasi-isomorphisms, but keeping it makes things more explicit].

- <u>Similarly</u>: $D^+(\mathcal{A}), D^-(\mathcal{A})$ (complexes bounded below, bounded above).

<u>Cones and triangles</u>:

- in category of top. spaces (or simplicial complexes etc.), $\nexists$ ker & coker !! (unless map is a fibration or an inclusion). However, <u>mapping cone</u> acts as both simultaneously:

$f : X \to Y \rightsquigarrow C_f := (X \times [0,1]) \amalg Y \Big/ \begin{matrix} (x,0) \sim (x',0) \\ (x,1) \sim f(x) \end{matrix}$



There are natural maps $Y \to C_f$ (inclusion) and $C_f \to \Sigma X$ (suspension) (collapse Y)

$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X \to \dots$

with compositions nullhomotopic, giving long exact sequence

$H_i(X) \to H_i(Y) \to H_i(C_f) \to \underset{\substack{\shortparallel \\ H_{i-1}(X)}}{H_i(\Sigma X)} \to \underset{\substack{\shortparallel \\ H_{i-1}(Y)}}{H_i(\Sigma Y)} \to \dots$

if X, Y simplicial complexes

$\rightsquigarrow C_f$ simplicial complex with i-cells = $\begin{cases} \text{cones on } (i-1)\text{ cells of } X \\ i\text{-cells of } Y \end{cases}$, $\partial = \begin{pmatrix} \partial_X & 0 \\ f & \partial_Y \end{pmatrix}$

③ By analogy: $f: A^\bullet \to B^\bullet$ chain map b/w complexes

$$\rightsquigarrow \quad C_f := A[1] \oplus B, \qquad d = \begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix}$$

ie $C_f^i = A^{i+1} \oplus B$

E.g: if A, B are single objects, $\mathrm{Cone}(f: A \to B)$ is just $\{A \xrightarrow{f} B\}$

We have natural chain maps $B^\bullet \xrightarrow{i} C_f^\bullet$ (inclusion of B as subcomplex)

$C_f^\bullet \xrightarrow{q} A^\bullet[1]$ (quotient complex)

(Can check $A^\bullet[1]$ is quasi-isomorphic to mapping cone of $i: B^\bullet \to C_f^\bullet$)

Thus, in derived category we don't have kernels & cokernels, but we have exact triangles

$$\begin{tikzcd} A^\bullet \arrow[r] & B^\bullet \arrow[dl] \\ C^\bullet \arrow[u, "{[1]}"] & \end{tikzcd}$$

or $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$

(with corresp. long exact seqs. in cohomology of complexes
$H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \dots$)

$\rightsquigarrow$ $D^b(\mathcal{A})$ is a triangulated category, namely additive cat. with a shift functor $T = [1]$ and a set of "distinguished triangles" satisfying various axioms, among which:

- $\forall X \in \mathrm{ob}$, $\begin{tikzcd} X \arrow[r, "id"] & X \arrow[dl] \\ 0 \arrow[u] & \end{tikzcd}$ is a distinguished triangle
- $\forall f: X \to Y$, $\exists$ dist. triangle $\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[dl] \\ C \arrow[u, "{[1]}"] & \end{tikzcd}$

---

* Back to Ext's & derived functors:

1) • The der. cat. gives a better way to understand derived functors.

Namely: $F: \mathcal{A} \to \mathcal{B}$ left exact functor b/w abelian categories

$\mathcal{R} \subset \mathcal{A}$ is an adapted class of objects if

- $\mathcal{R}$ is stable under direct sums
- $C^\bullet$ acyclic complex in $\mathcal{R}$ $\Rightarrow F(C^\bullet)$ acyclic
  $\hookrightarrow H^i(C) = 0$
- $\forall A \in \mathcal{A}$, $\exists$ inclusion $0 \to A \to R$, $R \in \mathcal{R}$.

(Ex: injectives)

$K^+(\mathcal{R})$ = homotopy category of complexes bounded below of objects in $\mathcal{R}$
morphisms = chain maps up to homotopy

Then: $RF :=$ composition $D^+(\mathcal{A}) \xrightarrow[\text{by elts of } \mathcal{R}]{\text{resolution}} K^+(\mathcal{R}) \xrightarrow[F]{} D^+(\mathcal{B})$

The functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is <u>exact</u>, ie. exact triangles $\mapsto$ exact triangles

Then $R^iF = H^i(RF)$ (What RF does for a single object $A \in \mathcal{A}$ is exactly what we do to compute $R^iF(A)$ using a resolution by objects of $\mathcal{R}$ & applying F, except taking cohomology).

2) Let $A, B \in \mathcal{A}$ (e.g. $Coh(X)$), view them as 1-step complexes in degree 0.

$B[k]$ shift ($B[k]^i = B^{i+k}$; so $B[k]$ concentrated in degree $-k$).

<u>Prop:</u> $\boxed{\mathrm{Hom}_{D^b(\mathcal{A})}(A, B[k]) = \mathrm{Ext}^k_{\mathcal{A}}(A, B)}$

* can use this to define product on $\mathrm{Ext}^k_{\mathcal{A}}(A,B) \otimes \mathrm{Ext}^\ell_{\mathcal{A}}(B,C) \to \mathrm{Ext}^{k+\ell}_{\mathcal{A}}(A,C)$ as composition in $D^b(\mathcal{A})$

<u>Example:</u> for $k=1$:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

no chain maps; but we're allowed to invert quasiisom's !!

If we have an extension $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ (s.e.s. in $\mathcal{A}$)

then we get maps of complexes

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "id"] & B \arrow[r] \arrow[u, "g"] & 0 \\ 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

quasi-isom.

which gives an element of $\mathrm{Hom}_{D^b(\mathcal{A})}(C, A[1])$ ($\simeq \mathrm{Ext}^1(C,A)$)
(can do the same with higher Ext's.)

* 2 ways to understand the proposition:

→ if $\mathcal{A}$ has enough injectives, take an injective resol^n of B and replace B by quasi-isom. complex (not bounded, but $D^b \hookrightarrow D^+$ is full and faithful...)

Then chain maps

$$\begin{tikzcd} & & A \arrow[d] & & \\ I_0 \arrow[r] & \cdots I_{k-1} \arrow[r] & I_k \arrow[r] & I_{k+1} \arrow[r] & \cdots \end{tikzcd}$$

up to homotopy $= H^k(\mathrm{Hom}(A, I_*))$.

→ check definition of Ext as derived functor:

say $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ s.e.s. in $\mathcal{A}$

Then get an exact triangle in $D^b(\mathcal{A})$: $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$

(w = extension map as above)

Axioms of triangulated categories ⇒

Prop.:
$$\begin{tikzcd} A \arrow[rr, "u"] & & B \arrow[dl, "v"] \\ & C \arrow[ul, "{[1]}", "w"'] & \end{tikzcd}$$
exact triangle, E object ⇒ long exact sequences

$$\dots \to \mathrm{Hom}(E, A[i]) \xrightarrow{u_*} \mathrm{Hom}(E, B[i]) \xrightarrow{v_*} \mathrm{Hom}(E, C[i]) \xrightarrow{w_*} \mathrm{Hom}(E, A[i+1]) \to \dots$$
$$\dots \to \mathrm{Hom}(A[i+1], E) \xrightarrow{w^*} \mathrm{Hom}(C[i], E) \xrightarrow{v^*} \mathrm{Hom}(B[i], E) \xrightarrow{u^*} \mathrm{Hom}(A[i], E) \to \dots$$

applying to our case (A, B, C, E 1-step complexes) we get exactly the defining property of Ext as derived functor of Hom ✓.

(Idea: e.g., exactness at $\mathrm{Hom}(E, B)$: (same at other places)

- check $vu = 0$ for any exact triangle:

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed, "\exists h"] & A[1] \arrow[d, "id"] \\ A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r] & A[1] \end{tikzcd}$$

axiom: $\exists h$ st. squares commute

h must be 0 ⇒ $vu = 0$ ✓

- now: assume $f: E \to B$ satisfies $vf = 0$.

$$\begin{tikzcd} E \arrow[r, "id"] \arrow[d, dashed, "\exists g"'] & E \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & 0 \arrow[r] \arrow[d, "0"] & E[1] \arrow[d, dashed] \\ A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & A[1] \end{tikzcd}$$

$\exists g$ st squares commute

⇒ $f = ug$

Hence $\mathrm{Ker}\, v_* = \mathrm{Im}\, u_*$ ✓).