① **Coherent sheaves (continued)** Ext groups: = right derived functor of Hom.

NB.: internal $\mathcal{H}om(\mathcal{E},\mathcal{F})$ = a sheaf
external $Hom(\mathcal{E},\mathcal{F})$ = global sections of $\mathcal{H}om$ = a vector space.

In general: a functor $F:\mathcal{C}\to\mathcal{C}'$ is left exact if
$0\to A\to B\to C\to 0$ short exact seq. $\Rightarrow$ $0\to F(A)\to F(B)\to F(C)$

Right derived functors: R^iF s.t.
$$0\to F(A)\to F(B)\to F(C)\to R^1F(A)\to R^1F(B)\to R^1F(C)\to R^2F(A)\to\dots$$

To compute $R^iF(A)$, resolve A by injective objects (injective: $Hom(-,I)$ exact)
($\to F$ becomes exact): $0\to A\to I^0\to I^1\to I^2\to\dots$

then get a complex $0\to F(I^0)\to F(I^1)\to F(I^2)\to\dots$

The cohomology of this complex gives $R^iF(A)=\dfrac{\ker(F(I^i)\to F(I^{i+1}))}{\mathrm{Im}(F(I^{i-1})\to F(I^i))}$

($R^0F(A)=F(A)$ by left exactness)

Example: sheaf cohomology = right derived functor of global sections

* $Hom(\mathcal{E},-)$ (covariant) and $Hom(-,\mathcal{F})$ (contravariant) are left-exact
$Ext^i=R^iHom$. In particular:

$0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0 \Rightarrow 0\to Hom(\mathcal{E},\mathcal{F}_1)\to Hom(\mathcal{E},\mathcal{F}_2)\to Hom(\mathcal{E},\mathcal{F}_3)$
$\to Ext^1(\mathcal{E},\mathcal{F}_1)\to Ext^1(\mathcal{E},\mathcal{F}_2)\to\dots$

$0\to\mathcal{E}_1\to\mathcal{E}_2\to\mathcal{E}_3\to 0\Rightarrow 0\to Hom(\mathcal{E}_3,\mathcal{F})\to Hom(\mathcal{E}_2,\mathcal{F})\to Hom(\mathcal{E}_1,\mathcal{F})$
$\to Ext^1(\mathcal{E}_3,\mathcal{F})\to Ext^1(\mathcal{E}_2,\mathcal{F})\to\dots$

In general, compute by resolving $\mathcal{F}$ by injectives (quasicoh., not coh.)

- since $Hom=H^0\mathcal{H}om$, could try to first understand failure of exactness of $\mathcal{H}om$, then that of global sections.

Fact: | if $\mathcal{E}$ is locally free (i.e. vect bundle) then $\mathcal{H}om(\mathcal{E},-)$ is exact
| Then $Ext^i(\mathcal{E},\mathcal{F})=H^i(\mathcal{H}om(\mathcal{E},\mathcal{F}))$.

Otherwise, resolve $\mathcal{E}$ by locally free sheaves
$0\to E_n\to\dots\to E_0\to\mathcal{E}\to 0$ then we can build a complex of sheaves $\mathcal{H}om(E_n,\mathcal{F})\leftarrow\mathcal{H}om(E_{n-1},\mathcal{F})\leftarrow\dots\leftarrow\mathcal{H}om(E_0,\mathcal{F})$
whose hypercohomology computes $Ext^i(\mathcal{E},\mathcal{F})$.

② Example: $\mathcal{E}$ loc. free (vector bundle)

$\mathcal{O}_p$ skyscraper sheaf at a point

- $\mathcal{H}om(\mathcal{E},\mathcal{O}_p) = \mathcal{E}^* \otimes \mathcal{O}_p =$ skyscraper sheaf with stalk $\mathcal{E}^*_{|p}$ at p.

  $\to \begin{cases} \mathrm{Hom}(\mathcal{E},\mathcal{O}_p) = \mathcal{E}^*_{|p} \\ \mathrm{Ext}^i(\mathcal{E},\mathcal{O}_p) = 0 \quad \forall i \geq 1 \end{cases}$ (skyscraper sheaves are acyclic)

- $\mathcal{H}om(\mathcal{O}_p,\mathcal{O}_p) = \mathcal{O}_p$ but this isn't the whole story...

  Resolve $\mathcal{O}_p$ by locally free sheaves, e.g. use Koszul resolution

  This is a local thing near $p \Rightarrow$ restricting, can assume X affine.

  Then local coords. near p define a section s of $V \simeq \mathcal{O}_X^{\oplus n}$ ($n = \dim$)

  $0 \to \Lambda^n V^* \xrightarrow{s} \Lambda^{n-1} V^* \xrightarrow{s} \cdots \to V^* \xrightarrow{s} \mathcal{O}_X \xrightarrow{s} \mathcal{O}_p \to 0$, apply $\mathcal{H}om(-,\mathcal{O}_p)$

  gives $\mathrm{Ext}^*(\mathcal{O}_p,\mathcal{O}_p)$ is hypercohomology of

  $$\mathcal{O}_p \xrightarrow[0]{} V\otimes\mathcal{O}_p \xrightarrow[0]{} \cdots \to \Lambda^{n-1}V\otimes\mathcal{O}_p \xrightarrow[0]{} \Lambda^n V\otimes \mathcal{O}_p$$

  ie. since complex is trivial, $\mathrm{Ext}^k(\mathcal{O}_p,\mathcal{O}_p) \simeq \Lambda^k V$.

- similarly, $\mathrm{Ext}^*(\mathcal{O}_p,\mathcal{E}) =$ hypercohomology of

  $$\mathcal{E} \xrightarrow[s]{} V\otimes\mathcal{E} \xrightarrow[s]{} \cdots \xrightarrow[s]{} \Lambda^{n-1}V\otimes\mathcal{E} \xrightarrow[s]{} \Lambda^n V\otimes\mathcal{E}$$

  can check this is exact except in last map, cokernel = skyscraper sheaf with stalk $(\Lambda^n V\otimes E)_{|p}$ at p. (in fact this is its Koszul resolution)

  Hence $\mathrm{Ext}^n(\mathcal{O}_p,\mathcal{E}) = \Lambda^n V\otimes E_{|p}$ ($\simeq E_{|p}$), all others zero.

  Consistent with Serre duality: $\mathrm{Ext}^i(\mathcal{E},\mathcal{F}) \cong \mathrm{Ext}^{n-i}(\mathcal{F}, K_X\otimes\mathcal{E})^\vee$

Derived categories: slogan: work with complexes up to homotopy.

★ enlarging a category to include complexes of objects makes it
- algebraically better behaved (e.g.: der. cat is triangulated)
- less sensitive to initial data (can restrict to nice subset of objects)
  (e.g.: on a smooth alg. var., coherent sheaves have a finite resolution by vector bundles, so can start with vector bundles instead of coherent sheaves ...)

(more important for Fukaya categories: allow immersed Lagrangians ?...)

③ ★ even if we know how to define general objects, it's usually easier to replace them by complexes of better-behaved objects.

E.g. $\mathcal{O}_D$, $D = s^{-1}(0)$, $s \in H^0(\mathcal{L})$ $\longleftrightarrow$ resolve by complex $\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X$

or Koszul resolution used above to compute Ext's for $\mathcal{O}_P$

<u>Another example</u>: intersection theory works better with complexes of nice objects

$D_1, D_2 \subset X$ smooth cx. surface defined by sections s_1, s_2 of $\mathcal{L}_1, \mathcal{L}_2$

$\to$ intersection theory: $[D_1]\cdot[D_2] = c_1(\mathcal{L}_1) \cup c_1(\mathcal{L}_2).[X] = c_1(\mathcal{L}_{1|D_2}).[D_2]$

If $D_1 \pitchfork D_2$ then $\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2} = \mathcal{O}_{D_1 \cap D_2}$ contains "the right information"

can also resolve by complex $\mathcal{L}^{-1}_{1|D_2} \xrightarrow{s_{1|D_2}} \mathcal{O}_{D_2}$ $\quad$ ($\mathrm{Coker} = \mathcal{O}_{D_1 \cap D_2}$)

(= apply $-\otimes \mathcal{O}_{D_2}$ to $\mathcal{L}_1^{-1} \xrightarrow{s_1} \mathcal{O}_X$)

But in non-transverse case, e.g. $D_1 = D_2 = D$, $\mathcal{O}_D \otimes \mathcal{O}_D = \mathcal{O}_D$ looks different?

Point: should instead work at level of complexes and apply $-\otimes \mathcal{O}_D$ to the resolution $\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X$ of $\mathcal{O}_D$, to get $\mathcal{L}^{-1}_{|D} \xrightarrow{s_{|D} = 0} \mathcal{O}_D$

Cokernel of $\mathcal{L}^{-1}_{|D} \xrightarrow{0} \mathcal{O}_D$ is still $\mathcal{O}_D$, but now there's also a kernel, which is the information we lost...

[information was lost because $-\otimes \mathcal{O}_D$ is only right exact, so $0 \to \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_D \to 0$ only yields $\mathcal{L}^{-1}_{|D} \xrightarrow{0} \mathcal{O}_D \to \mathcal{O}_D \to 0$. By contrast $-\otimes \mathcal{O}_D$ is exact on vector bundles]

★ <u>However</u>: a same object may have many different resolutions... when do we want to treat 2 complexes as isomorphic?

Looking at resolutions, it's tempting to think $H^*(\text{complex})$ is what we want, but this is much too coarse – loses important information.

<u>E.g.</u> <u>Whitehead</u>: X, Y simplicial complexes, simply connected:

then $X \underset{\text{h.e.}}{\simeq} Y$ iff. $\exists$ simplicial complex Z & maps $X \to Z \leftarrow Y$

s.t. chain maps $C^*(X) \leftarrow C^*(Z) \to C^*(Y)$ are isomo. on cohomology.

(whereas $H_*(X) \simeq H_*(Y)$ doesn't imply much, e.g. Massey products...)

(pass through Z = need to subdivide X/Y so homotopy equiv^{ce} between them can be approximated by a simplicial map)

Def: $C_\bullet \xrightarrow{f} D_\bullet$ chain map (i.e.
$$\begin{tikzcd} \cdots C_i \arrow[r, "\partial"] \arrow[d, "f"] & C_{i+1} \arrow[r, "\partial"] \arrow[d, "f"] & C_{i+2} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots D_i \arrow[r, "\partial"] & D_{i+1} \arrow[r, "\partial"] & D_{i+2} \arrow[r] & \cdots \end{tikzcd}$$
)

is a <u>quasiisomorphism</u> if the induced maps on cohomology are isomorphisms

This is stronger than $H^*(C_\bullet) \simeq H^*(D_\bullet)$

Ex: $\mathbb{C}[x,y]^2 \xrightarrow[(x,y)]{} \mathbb{C}[x,y]$ and $\mathbb{C}[x,y] \xrightarrow[0]{} \mathbb{C}$

not quasiisomorphic as complexes of $\mathbb{C}[x,y]$-modules even though same H^*

Ex: $\{\mathcal{L}^{-1} \xrightarrow[s]{} \mathcal{O}_X\}$ and $\mathcal{O}_D$ are quasiisomorphic, q-isom = cokernel map (similarly with other resolutions of coherent sheaves).