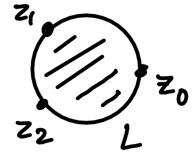
① **Recall:** approaches to $CF^*(L,L)$ and A_∞-algebra structure on it:

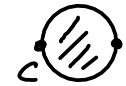
a) Hamiltonian perturbations (choose consistently! not strictly unital)
$CF^*(L,L) := \Lambda^{L \cap \varphi_H(L)}$

b) <u>FOOO</u>: $CF^*(L,L) = C_*(L,\Lambda)$ "chains on L"
m_k = count holom discs in (M,L) with incidence conditions at boundary marked pts

<u>E.g.</u>: product m_2 considers 

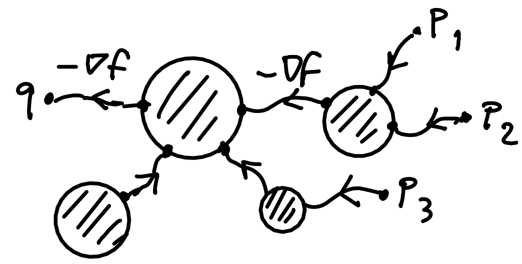
$ev_i : \bar{\mathcal{M}}_{0,3}(M,L;J,\beta) \to L$

$$m_2(c_2,c_1) = \sum_{\beta \in \pi_2(X,L)} ev_{0*}\left([\bar{\mathcal{M}}_{0,3}(M,L;J,\beta)] \cap ev_1^* c_1 \cap ev_2^* c_2\right) T^{\omega(\beta)}$$

(exception for m_1: don't count constant 

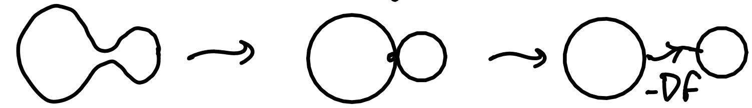
, instead ∂c as a chain)

c) <u>Cornea-Lalonde</u> approach to $CF^*(L,L)$ ("<u>clusters</u>")
fix a Morse function $f: L \to \mathbb{R}$, then $CF^*(L,L) = \Lambda^{\text{crit } f}$

m_k counts "clusters" of J-holom. discs + gradient flow lines 

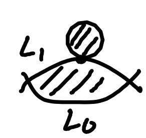
This cluster contributes to coeff^t of $T^{\text{area}} q$ in $m_3(p_1,p_2,p_3)$

Now bubbling of discs is no longer a boundary of moduli space:



Instead, broken Morse trajectories are boundaries ($\rightsquigarrow A_\infty$-eqns (?)).

<u>Discs and obstruction</u>:

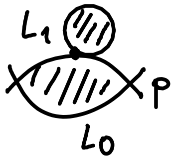
We've seen: if L_0 or L_1 bounds holom. discs then $\partial^2 \neq 0$ because index 2 moduli space has ends 

besides 

Count contributions of such discs $\rightsquigarrow m_0 \in CF^*(L,L)$

In FOOO's theory: $m_0 = \sum_{\beta \neq 0} ev_*\left[\bar{\mathcal{M}}_{0,1}(X,L,J,\beta)\right] T^{\omega(\beta)}$

②

Then 

is $m_2(m_0^{L_1}, p)$ and we get

$$m_1(m_1(p)) \pm m_2(\underbrace{m_0}_{\text{of } L_1}, p) \pm m_2(p, \underbrace{m_0}_{\text{of } L_0}) = 0$$

Hence, m_0 = obstruction to $\partial^2 = 0$

More generally, A_∞-equations hold if we include m_0 terms:

$\sum_{k,\ell \geq 0} \pm m_k(\dots, m_\ell(\dots), \dots) = 0$. This is called a "curved A_∞-category".

(& pretty hard to work with...)

Say L is unobstructed if $m_0 = 0$, weakly unobstructed if m_0 = mult. of 1.

($\Rightarrow$ central, so $m_1^2 = 0$ on $CF(L,L)$)

weakly unobstructed Lagrangians of a given "charge" form an honest A_∞-category

→ FOOO: try to cancel obstruction by deforming by $b \in CF^1(L,L)$:

on $CF^*(L,L)$, $m_k^b(c_k \dots c_1) = \sum m_{k+\ell}(b\dots b, c_k, b \dots b, c_{k-1}, \dots, c_1, b \dots b)$

still a curved A_∞-algebra; look for b s.t. $m_0^b = m_0 + m_1(b) + m_2(b,b) + \dots = 0$

or mult. of 1 so $(m_1^b)^2 = 0$: such b = "(weak) bounding cochain"

Then set objects = Lagrangians + equivalence classes of weak ∂ cochains

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③ <u>Coherent sheaves on a complex mfld:</u>

$\mathcal{O}_X$ sheaf of holomorphic functions

$\rightsquigarrow$ a coherent sheaf $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules

(ie. U open set $\mapsto \mathcal{F}(U)$ module $/ \mathcal{O}_X(U)$
w/ nice properties wrt restrictions ...)

s.t. (1) $\mathcal{F}$ is of finite type (ie. $\exists$ open cover of X by U's s.t.
$\mathcal{F}_{|U}$ is generated by a finite # of sections, ie. $\exists\ \mathcal{O}_{X|U}^{\oplus n} \twoheadrightarrow \mathcal{F}_{|U}$)

(2) $\forall U \underset{\text{open}}{\subset} X$, $\forall\ \phi: \mathcal{O}_{X|U}^{\oplus n} \to \mathcal{F}_{|U}$ hom. of $\mathcal{O}_X$-modules, $\ker(\varphi)$ is of finite type.

If X nice enough, $\Leftrightarrow$ $\mathcal{F}$ has <u>finite presentation</u> ie. $\exists$ open cover by subsets U st. $\exists$ exact seq. $\mathcal{O}_{X|U}^{\oplus m} \to \mathcal{O}_{X|U}^{\oplus n} \to \mathcal{F}_{|U} \to 0$.

ie. coherent sheaves are cokernels of morphisms of vector bundles

★ Main advantages over bundles: kernels & cokernels of morphisms of coherent sheaves are coherent sheaves.

<u>Ex:</u> • E vector bundle $\Rightarrow$ E (loc. free) sheaf (of holom. sections)

• D hypersurface defined by $s=0$, s section of L line bundle

$$\Rightarrow \quad 0 \to L^{-1} \xrightarrow[s]{} \mathcal{O}_X \longrightarrow \mathcal{O}_D \to 0$$

(for small enough U, s.e.s. on sections over U ✓).

• more generally, $Z \subset X$ codim. r subvar. defined transversely as zero set of $s \in H^0(E)$ E rank r v.b.

$\Rightarrow$ Koszul resolution: $0 \to \Lambda^r E^* \xrightarrow[s]{} \Lambda^{r-1} E^* \xrightarrow[s]{} \cdots \to E^* \xrightarrow[s]{} \mathcal{O}_X \to \mathcal{O}_Z \to 0$

★ X smooth $\Rightarrow$ coherent sheaves always have a finite resolution by vector bundles.

★ <u>Ext groups:</u> = right derived functor of Hom.

<u>NB:</u> internal $\mathcal{H}om(\mathcal{E},\mathcal{F})$ = a sheaf

external $\mathrm{Hom}(\mathcal{E},\mathcal{F})$ = global sections of $\mathcal{H}om$ = a vector space.

In general: a functor $F: \mathcal{C} \to \mathcal{C}'$ is <u>left exact</u> if

$0 \to A \to B \to C \to 0$ short exact seq. $\Rightarrow$ $0 \to F(A) \to F(B) \to F(C)$

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* Right derived functors: R^iF s.t.

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots$$

To compute, $R^iF(A)$, resolve A by injective objects (injective: $\mathrm{Hom}(-,I)$ exact)
($\to F$ becomes exact): $0 \to A \to I^0 \to I^1 \to I^2 \to \dots$

then get a complex $0 \to F(I^0) \to F(I^1) \to F(I^2) \to \dots$

The cohomology of this complex gives $R^iF(A) = \dfrac{\ker(F(I^i) \to F(I^{i+1}))}{\mathrm{Im}(F(I^{i-1}) \to F(I^i))}$

($R^0F(A) = F(A)$ by left exactness)

<u>Example:</u> sheaf cohomology = right derived functor of global sections

Namely, ses of sheaves $\Rightarrow 0 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to \dots$

compute by resolving by acyclic sheaves (e.g. flasque sheaves ...)
(coincides with Čech cohomology)