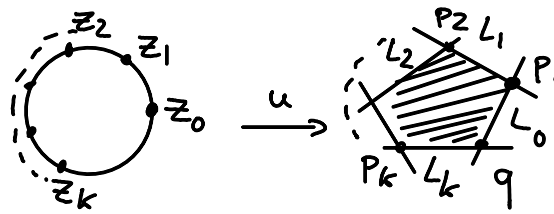
① **Recall:** Floer complexes $CF^*(L,L') = \Lambda^{|L\cap L'|}$ carry differential m_1, product m_2, & higher operations $CF^*(L_0,L_1)\otimes\cdots\otimes CF^*(L_{k-1},L_k) \xrightarrow{m_k} CF^*(L_0,L_k)[2-k]$

grading shift.

Look at J-hol. maps



D^2 with $(k+1)$ boundary marked pts

$$\text{exp. dim } \mathcal{M}(p_1\cdots p_k, q, [u], J) = \deg q - (\deg p_1 + \cdots + \deg p_k) + k - 2$$

Assuming transversality, $m_k(p_k\cdots p_1) := \displaystyle\sum_{\substack{q\in L_0\cap L_k \\ [u]/\,\mathrm{ind}=0}} \left(\#\mathcal{M}(p_1\cdots p_k, q, [u], J)\right) T^{\omega(u)}\, q$

get <u>A_∞-relations</u> when consider ∂ of 1-dim! families of discs.

<u>Prop:</u> Assuming no bubbling of discs/spheres, we have $\forall m\geq 1$, $\forall p_i \in L_{i-1}\cap L_i$,

$$\sum_{\substack{k,\ell\geq 1 \\ k+\ell = m+1 \\ 0\leq j\leq \ell-1}} (-1)^* m_\ell(p_m,\ldots,p_{j+k+1}, m_k(p_{j+k},\ldots,p_{j+1}), p_j,\ldots,p_1) = 0$$

where $* = \deg(p_1)+\cdots+\deg(p_j) + j$

$\rightsquigarrow$ m_1 is a differential
m_2 satisfies Leibniz rule wrt m_1
m_2 is associative up to homotopy given by m_3
- - - -

<u>Def:</u> <u>A_∞-category</u> = linear "category" (except associativity...) where morphism spaces are equipped with such algebraic operations $(m_k)_{k\geq 1}$

<u>Fukaya category</u> = A_∞-cat. with objects = Lagrangians
morphisms = Floer complexes
alg operations = as above.

(many versions with different details – we'll see later).

So far we have at best an A_∞-<u>precategory</u> ie. morphisms and compositions are defined only for transverse objects.
$(CF(L,L) = ??)$

② * At the homology level, the Donaldson-Fukaya category (hom = HF) is easier to work with but contains less information in general!

* "Convergent power series" Floer homology:

We've recorded holom. disks with weights $T^{\omega(u)}$.

Gromov compactness $\Rightarrow$ Σ may be infinite but well-def! in Novikov ring Λ

Physicists would actually write $e^{-2\pi\omega(u)} \in \mathbb{R}$ and hope for convergence.

Working over Λ, from a physicist's perspective, amounts to considering a family of symplectic forms $(M, \omega_t = t\omega)$ $(\leftrightarrow T = e^{-2\pi t})$ near the large volume limit $(t \to \infty)$ and computing Floer homologies for all ω_t simultaneously (for t large, if radius of convergence is nonzero; or purely as a formal family near large vol. limit).

Beware: even when it is defined, convergent power series HF^* need not be a Hamiltonian isotopy invariant.

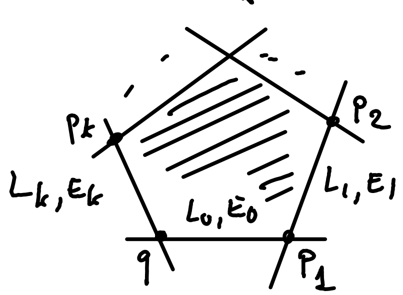
* Twisted coefficients:

L_i Lagrangians are equipped with $(E_i, \nabla_i) \to L_i$ vector bundles w/ flat connections (think of: $\mathbb{C}$ vect. bundle w/ flat unitary conn., but could generalize to Novikov).

Define $CF((L_0, E_0, \nabla_0), (L_1, E_1, \nabla_1)) = \bigoplus_{p \in L_0 \cap L_1} \mathrm{Hom}((E_0)_p, (E_1)_p) \otimes \Lambda$

Then given $p_1 \cdots p_k$ $(p_i \in L_{i-1} \cap L_i)$ and $w_i \in \mathrm{Hom}((E_{i-1})_{p_i}, (E_i)_{p_i})$,

let $m_k(w_k \cdots w_1) := \sum_{\substack{q \in L_0 \cap L_k \\ [u]}} (\#\mathcal{M}(p_1 \cdots p_k, q, [u], J))\, T^{\omega([u])} \underbrace{\mathcal{P}_{[\partial u]}(w_k \cdots w_1)}_{\in \mathrm{Hom}((E_0)_q, (E_k)_q)}$



parallel transport along ∂u from q to p_1 using ∇_0 gives $\gamma_0 \in \mathrm{Hom}((E_0)_q, (E_0)_{p_1})$

p_i	p_{i+1}	∇_i	$\gamma_i \in \mathrm{Hom}((E_i)_{p_i}, (E_i)_{p_{i+1}})$
p_k	q	∇_k	

③ Flatness of D_i ⇒ these depend only on homotopy class of u.

$$\rightsquigarrow \rho_{[\partial u]}(w_k \cdots w_1) := \gamma_k \circ w_k \circ \gamma_{k-1} \circ \cdots \circ \gamma_1 \circ w_1 \circ \gamma_0 \in \mathrm{Hom}((E_0)_q, (E_k)_q)$$

Esp. important to us: E_i = top. trivial line bundle $\mathbb{C} \times L_i$
D_i = flat $U(1)$ connection $D_i = d + iA_i$, A_i closed 1-form

Then • $CF = \bigoplus_{p \in L_0 \cap L_1} \Lambda_{\mathbb{C}}\, p$

• (generator: p, $w = Id : E_{0_p} \simeq E_{1_p}$)

⇒ m_k counts discs with weights $T^{\omega(u)}\, hol(\partial u)$

where $hol(\partial u) \in U(1)$ = holonomy for parallel transport around loop ∂u, defined using identification at corners $= \exp\left(i \sum_{j=0}^{k} \int_{(\partial u)_j} A_j\right)$

• First iteration of Fukaya category (as an A_∞-precat.)
  - objects = $\mathcal{L} = (L, E, \nabla)$,
    L compact spin Lagrangian ($\mathbb{Z}$-graded version: $\mu_L = 0$, + grading data) s.t. L doesn't bound holom. discs.
    (E, ∇) flat hermitian vector bundle
  - for $\mathcal{L}_0 \pitchfork \mathcal{L}_1$, $hom(\mathcal{L}_0, \mathcal{L}_1) := CF^*$ Floer complex
  - for transverse sequence, m_k = operations on Floer complex.

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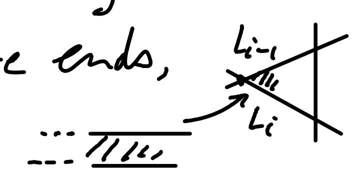
2 major outstanding issues: • L_0 not transverse to L_1? • L bounds discs?

1) what to do if L_0, L_1 not transverse? in particular, $CF^*(L, L)$?

Various approaches in literature:

a) pick a Hamiltonian perturbation to make them transverse.
(i.e., define $CF^*(L_0, L_1)$ to be generated by $L_0 \cap \varphi_H(L_1)$ where $H = H(L_0, L_1)$, and perturb all holom. curve equations by suitable Hamiltonian terms — in particular, in strip like ends,
want $H \to H(L_{i-1}, L_i)$)



See e.g. Seidel's book.

(4)

Main issues: (at chain level! Floer homology easier...)

- need to fix consistent choices of perturbation data.

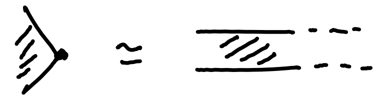
  (need a procedure which associates to each pair (L,L') a Hamiltonian $H(L,L')$, and to each sequence $(L_0 \dots L_k)$, perturbation data for $(k+1)$-marked holom. discs s.t. converges to $H(L_{i-1}, L_i)$ in each strip-like end)

  + show different choices yield equivalent categories

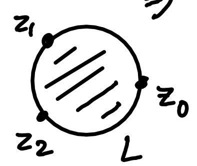
- no canonical strict unit $1 \in CF^*(L,L)$. (only a homology unit)

b) "Morse-Bott" Floer homology (e.g. FOOO)

- $CF^*(L,L) := C_*(L;\Lambda)$ "singular chains" on L (in a suitable sense...)

  Operations m_k: instead of a strip-like end 

  put a boundary marked point z and require $u(z) \in$ Chain.

  <u>E.g.</u>: product m_2 considers 

  $ev_i : \bar{\mathcal{M}}_{0,3}(X,L;J,\beta) \to L$

  $$m_2(C_2,C_1) = \sum_{\beta \in \pi_2(X,L)} ev_{0*}\left([\bar{\mathcal{M}}_{0,3}(X,L;J,\beta)] \cap ev_1^* C_1 \cap ev_2^* C_2\right) T^{\omega(\beta)}$$

  contribution of constant discs $\equiv$ intersection product $C_*(L)$

  (exception for m_1: don't count constant , instead ∂C as a chain)

- more generally, if L_0, L_1 have "clean intersection", ie $L_0 \cap L_1$ smooth and L_0, L_1 transverse in normal direction to $L_0 \cap L_1$, want to set $CF^*(L_0,L_1) = C_*(L_0 \cap L_1; \Lambda)$ & use chain as incidence condition at strip-like end — analytical details not completely clear.