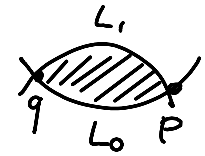
1 **Recall:**

$L_0, L_1 \subset (M,\omega)$ transverse Lagrangians $\to CF(L_0,L_1) = \Lambda^{|L_0\cap L_1|}$

with differential $\quad \partial(p) = \displaystyle\sum_{\substack{q\in L_0\cap L_1 \\ \phi\in\pi_2 / \operatorname{ind}(\phi)=1}} \left(\#\mathcal{M}(p,q,\phi,J)/_{\mathbb{R}}\right) T^{\omega(\phi)}\, q$

where $\mathcal{M} = \left\{ \begin{array}{l} \text{finite energy } J\text{-hol. maps } \; u:\mathbb{R}\times[0,1]\to M \\ u(s,0)\in L_0,\ u(s,1)\in L_1,\ \lim\limits_{s\to+\infty} u = p,\ \lim\limits_{s\to-\infty} u = q \end{array} \right\}$

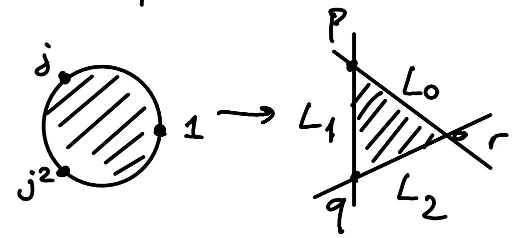


__Product structure__: $\quad CF^*(L_0,L_1)\otimes CF^*(L_1,L_2) \to CF^*(L_0,L_2)$

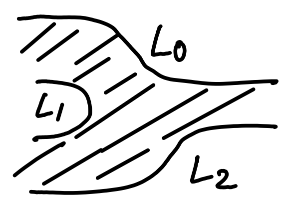
Look at $\; u: D^2 \to M \;$ J-holom. disk with

$u(j) = p\in L_0\cap L_1,\ u(j^2) = q\in L_1\cap L_2,\ u(1) = r\in L_0\cap L_2$

$u([1,j])\subset L_0,\quad u([j,j^2])\subset L_1,\quad u([j^2,1])\subset L_2$



(or equivalently, u:



$\to M$ Riem. surface of genus 0 with 3 strip-like ends [of finite energy])

Let $\mathcal{M}(p,q,r,[u],J) = \{\text{ such maps }\}$

expected dim. $= \operatorname{ind}([u]) = \deg r - (\deg p + \deg q)$

(where trivialize u^*TM & pick graded Lfts to define the degrees)

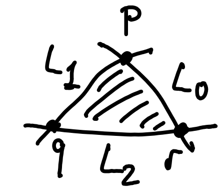
Then set $\quad \left\| \; q\cdot p = \displaystyle\sum_{\substack{r\in L_0\cap L_2 \\ \phi\in\pi_2/\operatorname{ind}(\phi)=0}} \left(\#\mathcal{M}(p,q,r,\phi,J)\right) T^{\omega(\phi)}\, r \right.$

__Notes:__
- $\to$ as usual, this is subject to achieving transversality, orientability ...
- $\to$ $\operatorname{Aut}(D^2)$ acts transitively on cyclically ordered triples of boundary points, so choice of $(1,j,j^2)$ is arbitrary.
- $\to$ lack of symmetry in $\deg p,q,r$ of index formula is because the degree of $r\in CF(L_0,L_2)$ is n minus that of $r\in CF(L_2,L_0)$

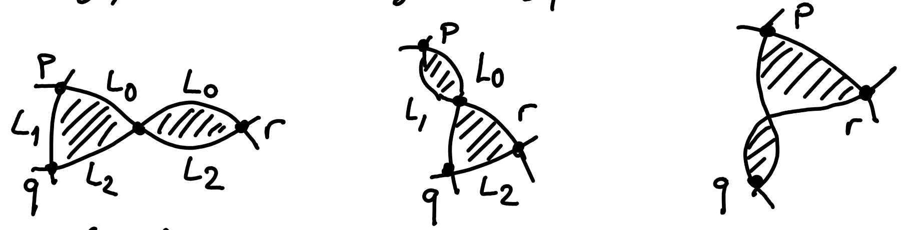
In general we have a "Poincaré duality" $CF^*(L,L') \simeq CF^{n-*}(L',L)^\vee$, compatible with differential, product, ...

② **Prop:** If $[\omega].\pi_2(M, L_i) = 0$ then the product satisfies Leibniz rule wrt differential, and hence induces a product on HF^*.
Moreover, the product on HF^* is associative.

Idea pf: (1) for Leibniz rule: consider index 1 moduli spaces



compactify by adding limit configurations: in the absence of bubbling, those are of 3 types:



$\partial(q \cdot p)$ $\qquad$ $q \cdot (\partial p)$ $\qquad$ $(\partial q) \cdot p$

gluing theorem: assuming transversality, adding these gives a 1-manifold with boundary.

#ends $= 0$ (w/ orientations, or mod 2) $\Rightarrow$ Leibniz identity.
(w/ signs depending on degree)

Thus:
- p, q closed $\Rightarrow \partial(q \cdot p) = \pm(\partial q) \cdot p \pm q \cdot (\partial p) = 0$
- ∂p exact, q closed $\Rightarrow q \cdot \partial p = \pm \partial(q \cdot p) \pm \underbrace{(\partial q)}_{0} \cdot p$ exact.

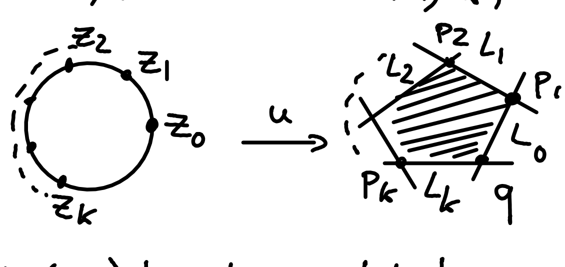
$\rightsquigarrow$ get product on HF^*

(2) associativity: we'll see now. ◀

Higher operations: $CF^*(L_0, L_1) \otimes \dots \otimes CF^*(L_{k-1}, L_k) \xrightarrow{m_k} CF^*(L_0, L_k)[2-k]$

grading shift.

Look at J-hol. maps



D^2 with $(k+1)$ boundary marked pts
(Riem. surface w/ boundary, with $(k+1)$ strip-like ends)

exp. dim $\mathcal{M}(p_1 \dots p_k, q, [u], J) = \deg q - (\deg p_1 + \dots + \deg p_k) + k - 2$

(3) The term $k-2$ comes from the dim. of the moduli space of discs with $k+1$ marked points. Assume we can achieve transversality:

Then $$m_k(p_k \dots p_1) := \sum_{\substack{q \in L_0 \cap L_k \\ [u]/\, ind = 0}} \left(\# \mathcal{M}(p_1 \dots p_k, q, [u], J)\right) T^{\omega(\phi)} q$$

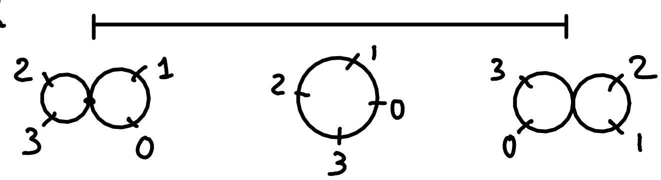
(m_1 = differential, m_2 = product).

NB: moduli space of discs with $(k+1)$ boundary marked points:

$\mathcal{M}_{0,k+1} = \{(z_0, \dots, z_k) \in \partial D^2 \text{ distinct, in order}\}$ contractible, dim. $k-2$

compactifies to moduli space $\overline{\mathcal{M}}_{0,k+1}$ of stable genus 0 Riem surf. w/ one ∂ component & $k+1$ boundary marked pts, ie. trees of discs attached together at marked nodal points, s.t. each component has ≥ 3 special points

E.g: $\overline{\mathcal{M}_{0,4}}$ = closed interval



$\Rightarrow$ when considering sequences of holom. discs as above, limit configurations allowed by Gromov compactness =

- bubbling of spheres, of discs
- breaking of strips at marked pts
- degeneration of domain to $\partial \overline{\mathcal{M}}_{0,k+1}$

} (energy accumulates at various places in domain)

get relations when consider ∂ of 1-dim! families of discs.

Prop: Assuming no bubbling of discs/spheres, we have $\forall m \geq 1$, $\forall p_i \in L_{i-1} \cap L_i$,

$$\sum_{\substack{k,\ell \geq 1 \\ k+\ell = m+1 \\ 0 \leq j \leq \ell - 1}} (-1)^{*} m_\ell(p_m, \dots, p_{j+k+1}, m_k(p_{j+k}, \dots, p_{j+1}), p_j, \dots, p_1) = 0$$

where $* = \deg(p_1) + \dots + \deg(p_j) + j$

Ex: $\underset{\text{differential}}{m_1(m_1(p)) = 0}$; $\quad \underset{\text{Leibniz rule}}{m_1(m_2(p,q)) + m_2(p, m_1(q)) + (-1)^{\deg q + 1} m_2(m_1(p), q)}$

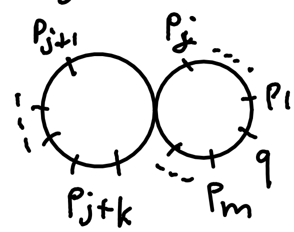
④

next one: $m_1(m_3(p,q,r)) \pm m_2(m_2(p,q),r) \pm m_2(p, m_2(q,r))$
$\pm m_3(m_1(p),q,r) \pm m_3(p,m_1(q),r) \pm m_3(p,q,m_1(r)) = 0$

says: the product m_2 is associative up to homotopy
(the homotopy being given by m_3).
& hence associative on cohomology.

and so on.

<u>Idea pf:</u> consider a 1-dim^l moduli space $\mathcal{M}(p_1 \dots p_m, q; [u], J)$ and its ends:
Assuming transversality & absence of bubbling, limiting configs are all of the form



(these are the codim-1 strata; configs with more components have higher codimension).

Total # ends $= 0 =$ sum of terms in the proposition
(coeff^t of $T^{\omega([u])} q$ in $\Sigma \dots$) *

<u>Def:</u> A_∞<u>-category</u> = linear "category" (except associativity...) where morphism spaces are equipped with such algebraic operations $(m_k)_{k \geq 1}$

<u>Fukaya category</u> = A_∞-cat. with objects = Lagrangians
morphisms = Floer complexes
alg operations = as above.

(many versions with different details – we'll see later).

So far we have at best an A_∞-<u>precategory</u> i.e. morphisms and compositions are defined only for transverse objects.
($CF(L,L) = ??$)

* At the homology level, the Donaldson-Fukaya category (hom = HF) is easier to work with but contains less information in general!