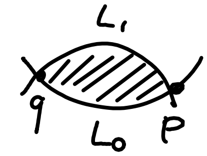
① $L_0, L_1 \subset (M,\omega)$ transverse Lagrangians $\to CF(L_0,L_1) = \Lambda^{|L_0 \cap L_1|}$

with differential $\displaystyle \partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ \phi \in \pi_2 / \operatorname{ind}(\phi)=1}} \left(\# \mathcal{M}(p,q,\phi,J)/\mathbb{R}\right) T^{\omega(\phi)}\, q$

where $\mathcal{M} = \left\{ \begin{array}{l} \text{finite energy } J\text{-hol. maps } u : \mathbb{R}\times[0,1] \to M \\ u(s,0) \in L_0,\ u(s,1) \in L_1,\ \lim_{s\to+\infty} u = p,\ \lim_{s\to-\infty} u = q \end{array} \right\}$



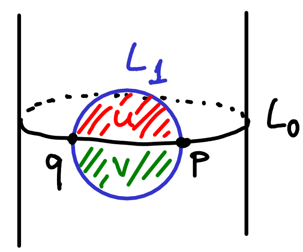
Limits of sequences in $\mathcal{M}$ have
- sphere bubbling (codim ≥ 2 if transv)
- disc bubbling
- broken strips

} codim 1 if transv.

We've seen: if there is no bubbling (e.g. if $\omega \cdot \pi_2(M, L_i) = 0$) then $\partial^2 = 0$ (by considering ends of moduli spaces of index 2 strips).

☆ Bubbling of discs is not just a technical issue to overcome, it's an actual obstruction to defining Floer homology

<u>Example:</u> $T^*S^1 \supset$

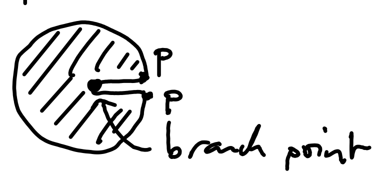


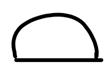
$CF(L_0, L_1) = \Lambda p \oplus \Lambda q$

$\partial p = \pm T^{\text{area}(u)}\, q$

$\partial q = \pm T^{\text{area}(v)}\, p$

so... $\partial^2 \neq 0$! what goes wrong: look at moduli space of index 2 strips from p to itself. It's an interval...



(parametrizing by upper half-disc ,
& setting up L_1 = unit circle, L_0 = real axis,
these are: $u_\alpha(z) = \dfrac{z^2+\alpha}{1+\alpha z^2}$, $\alpha \in (-1,1)$)

The two end points: 1)

broken trajectory $p \to q \to p$
(contributing to $\partial^2 p$)

2)

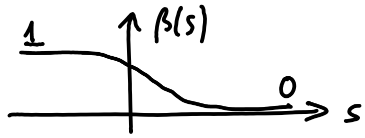
constant strip at p
+ disc bubble with boundary in L_1

So the disc bubble prevents $\partial^2 = 0$ ...

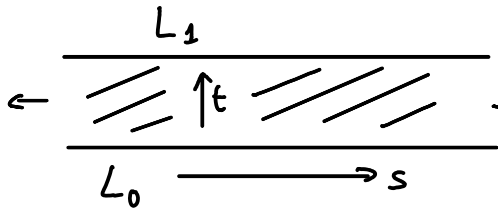
② * <u>Hamiltonian isotopy inv^{ce}</u>: say $H : [0,1]\times M \to \mathbb{R}$ generates ϕ_H^t = flow of X_H $(\iota_{X_H}\omega = dH)$

Consider finite energy solutions of
$$\begin{cases} u : \mathbb{R}\times[0,1] \longrightarrow M \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - \beta(s)\, X_H(t,u)\right) = 0 \\ u(s,0)\in L_0, \quad u(s,1)\in L_1 \end{cases}$$

where β = cutoff function



cv to trajectory
$\dot\gamma(t) = X_H(t,\gamma)$
$\gamma(0)\in L_0$, $\gamma(1)\in L_1$ ←



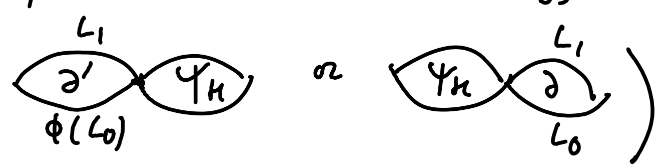
→ cv to $p\in L_0\cap L_1$

$\Leftrightarrow \gamma(1) = q \in \phi_H^1(L_0)\cap L_1$

(set $\tilde u(s,t) = \phi_H^{(t,1)}(u(s,t))$ then satisfies unperturbed $\bar\partial_J$-eq^n for $s \ll 0$).

Counting <u>index 0</u> solutions gives $\Psi_H : CF(L_0,L_1) \to CF(\phi_H^1(L_0), L_1)$
(isolated; no $\mathbb{R}$-transl. inv^{ce} now!)

In absence of disc bubbling, can show this is a <u>chain map</u> $\Psi_H\circ\partial = \partial'\circ\Psi_H$.
(<u>idea</u>: look at ends of index 1 moduli spaces = if no disc bubbling, they must be broken trajectories



)

+ this chain map induces an isomorphism in homology.
(idea: look at Ψ_H and Ψ_{-H} for reversed isotopy, then build a homotopy between $\Psi_H\circ\Psi_{-H}$ and Id ....).

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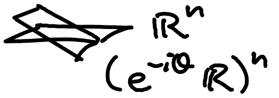
* More about grading: want gradings on $CF(L_0,L_1)$ st. $\deg(q)-\deg(p) = \text{index}$?
Recall Maslov index $\Leftrightarrow \pi_1(\Lambda Gr) = \mathbb{Z}$. Things are easier if $c_1(M)=0$ (or trivial on π_2)
then ΛGr-bundle of Lagr. planes over M admits a fiber-wise universal cover — $\widetilde{\Lambda Gr}$-bundle of "graded Lagr. planes". Then, if at p we fix graded lifts of T_pL_i we can define the Maslov index of the intersection at p.

If L_1 is slightly clockwise from L_0, 

then set $\deg(p) = 0$

otherwise, set $\deg(p)$ = Maslov index from this reference configuration.

Obstruction to defining <u>globally</u> graded HF of L := <u>Maslov class</u> $\mu_L \in H^1(L, \mathbb{Z})$. If it vanishes then $\mathrm{ind}(u) = \deg(q) - \deg(p)$ depends only on p, q, not on the homotopy class $[u]$ $\Rightarrow$ Floer homology is $\mathbb{Z}$-graded.

Otherwise HF is only $\mathbb{Z}/N$-graded, N = minimal Maslov number

... or can get $\mathbb{Z}$-graded theory by working over a larger ring, with an extra generator to keep track of Maslov index [in <u>monotone</u> case, where area & Maslov index are proportional, can just set $\deg(T) \neq 0$].

* <u>Note</u>: if L_i are oriented, then grading mod 2 $\equiv$ sign of intersection

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<u>Example</u>: $M = T^*N$, $\omega = \sum dp_i \wedge dq_i$

Equip N with a Riem. metric g, induces metric & a.c.s. on T^*N

(along zero section, $TM = TN \oplus T^*N$, $T^*N \simeq TN$ via g

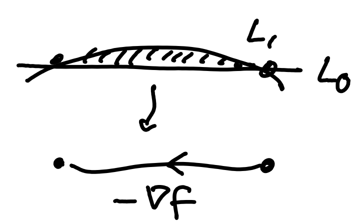
Then $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$)

L_0 = zero section, $L_1 = \mathrm{graph}(\varepsilon\, df)$, f Morse function on N, ε small

- $L_0 \cap L_1 = \mathrm{crit}(f)$

  Maslov index $\Leftrightarrow$ n − Morse index of crit pt

- (Fukaya-Oh) For $\varepsilon \to 0$, holom. strips $\overset{1-1}{\longleftrightarrow}$ gradient flow trajectories



$\Rightarrow HF^*(L_0, L_1) \simeq HM_{n-*}(f) \;(\simeq H^*(N))$

(can discard Novikov coeffts as all strips $p \to q$ have $\int u^*\omega = \varepsilon(f(q) - f(p))$)

* by Weinstein nbd thm, this is a univ local model for $L \subset M$ & a C^1-small Hamiltonian deformation of L. By Ham. isotopy inv^ce of HF, set $HF(L,L) := HF(L, \varphi(L))$. If L doesn't bound discs we conclude $HF(L,L) \cong H^*(L)$

(4) If L does bound discs, but under a suitable assumption to ensure HF well-defined, e.g. L monotone ie. ω and Maslov positively proportional on $\pi_2(M, L)$, we have a filtration of Floer complex & a spectral sequence starting with $H^*(L; \Lambda)$ and converging to $HF(L, L)$ (with successive diff!s = contributions of holom. discs of increasing area). $\rightarrow$ Oh spectral sequence.