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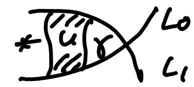
Lagrangian Floer homology:

(M,ω) symplectic manifold $\supset L_0, L_1$ compact Lagrangian submanifolds

Formally, Floer homology = Morse theory for "action functional" on path space $\mathcal{P}(L_0,L_1)$, whose crit pts are constant paths & gradient flowlines = J-hol. strips.

More precisely: $\mathcal{A} : \underset{\text{univ. cover}}{\widetilde{\mathcal{P}}(L_0,L_1)} \to \mathbb{R}$, $(\gamma, [u])$  $\mapsto \int u^*\omega$

$u : [0,1]^2 \to M$ homotopy $* \to \gamma$

$$d\mathcal{A}(\gamma)\cdot v = \int_{[0,1]} \omega(\dot\gamma, v)\,dt = \int_{[0,1]} g(J\dot\gamma, v)\,dt = \langle J\dot\gamma, v\rangle_{L^2}$$

v vect. field along γ, $v(0) \in T_{\gamma(0)}L_0$, $v(1) \in T_{\gamma(1)}L_1$

hence crit pts = const. paths $\dot\gamma = 0$; gradient trajs. = J-hol. maps $\frac{\partial\gamma}{\partial s} = -J\dot\gamma$

Difficult to define rigorously ∞-dim! Morse theory, so use holom. curves instead.

Actual setup: Assume $L_0 \pitchfork L_1 \Rightarrow L_0 \cap L_1$ finite set.

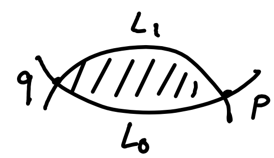
Recall Novikov ring $\Lambda = \{\sum a_i T^{\lambda_i} \mid \lambda_i \to +\infty\}$

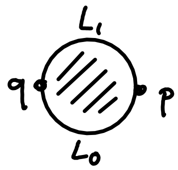
Floer complex $CF(L_0,L_1) = \Lambda^{|L_0\cap L_1|}$ free Λ-module gen^d by $L_0 \cap L_1$.

Goal: define a differential ∂ by counting holomorphic discs:

Look at: $u : \mathbb{R}\times[0,1] \longrightarrow M$ equipped with J ω-compat. a.c.s.

s.t. $(*)$
- $\bar\partial_J u = 0$, ie. $\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$
- $u(s,0) \in L_0$, $u(s,1) \in L_1$
- $\lim_{s\to+\infty} u(s,t) = p$, $\lim_{s\to-\infty} u(s,t) = q$
- the energy $E(u) = \int u^*\omega = \iint \left|\frac{\partial u}{\partial s}\right|^2 < \infty$



(NB: $\mathbb{R}\times[0,1] \underset{\text{biholom.}}{\simeq} D^2 \setminus \{\pm 1\}$ so also think of maps 

)

$\mathcal{M}(p,q,[u],J) = \{u \text{ sol}^{ns} \text{ of } (*)\}$ moduli space of holom. discs.

$[u] \in \pi_2(M; L_0, L_1)$ homotopy class

$(*)$ is a Fredholm problem, exp. dim. $\mathcal{M} = \operatorname{ind}(\bar\partial_J)$

$\operatorname{ind}(\bar\partial_J) =$ <u>Maslov index</u>

comes from $\pi_1(\Lambda Gr) = \mathbb{Z}$ for Lagrangian grassmannian in $\mathbb{R}^{2n}$.

② Maslov index:

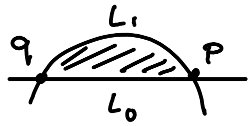
Let L_0, $L_1(t)_{t\in[0,1]}$ Lagr. subspaces of $\mathbb{R}^{2n}$, s.t. $L_1(0), L_1(1) \pitchfork L_0$

Then Maslov index of the path $L_1(t)$:= # times that $L_1(t)$ fails to be transverse to L_0 (counted with signs & multiplicities)

Ex. path $(e^{i\theta_1}\mathbb{R})\times\dots\times(e^{i\theta_n}\mathbb{R})$ if $\theta_i \nearrow$ through 0, then $\mu(L_0, L_1(t)) = n$.

Now: given a strip u, trivialize $u^*TM \rightsquigarrow u^*TL_0, TL_1$ paths of Lagrangians. Can trivialize so that TL_0 remains constant

Then $\mathrm{ind}(u) :=$ Maslov index of path TL_1 relative to TL_0 as one goes from p to q

Ex: 

in $\mathbb{R}^2$ has $\mathrm{ind}([u]) = 1$.

• Want to define: $$\partial(p) = \sum_{\substack{q\in L_0\cap L_1 \\ \phi\in\pi_2 /\, \mathrm{ind}(\phi)=1}} \left(\#\mathcal{M}(p,q,\phi,J)/\mathbb{R}\right) T^{\omega(\phi)}\, q$$

symp. area of ϕ $(=\int u^*\omega)$; translation

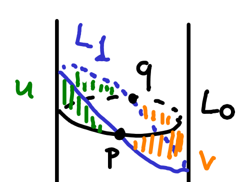
Issues:
- transversality
- compactness, bubbling
- orientation of $\mathcal{M}$ ($\Rightarrow$ signed counts? else work over $\mathbb{Z}/2$)
- $\partial^2 = 0$?

A special case where bubbling isn't an issue:

Thm (Floer): if $[\omega]\cdot\pi_2(M) = 0$ and $[\omega]\cdot\pi_2(M, L_i) = 0$ then ∂ is well-def^d, $\partial^2 = 0$, and HF is indep^t of chosen J & invariant under Hamiltonian deformations of L_0 and/or L_1

Corollary: $[\omega]\cdot\pi_2(M,L) = 0$, ψ Ham. diffeo, $\psi(L) \pitchfork L \Rightarrow |\psi(L)\cap L| \geq \sum b_i(L)$

(special case of Arnold's conjecture; idea: $HF(L,\psi(L)) \simeq H^*(L)$; rank CF $\geq$ rank HF)

Example: $T^*S^1 = \mathbb{R}\times S^1$ 

$CF(L_0, L_1) = \Lambda p \oplus \Lambda q$

$\partial p = (T^{\mathrm{area}(u)} - T^{\mathrm{area}(v)})\, q$

$\partial q = 0$.

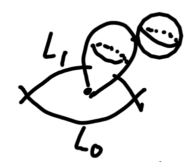
③ In this case $\exists$ well-def^{d} $\mathbb{Z}$-grading (Maslov index only depends on p, q, not on homotopy class of strip), e.g. $\deg(p) = 0$, $\deg(q) = 1$.

2 cases:
- $\mathrm{area}(u) = \mathrm{area}(v)$ (L_0, L_1 Ham.-isotopic): $HF(L_0, L_1) \simeq H^*(S^1; \Lambda)$
- $\mathrm{area}(u) \neq \mathrm{area}(v)$ (L_0, L_1 can be disjoined): $HF(L_0, L_1) = 0$.

Back to issues with the definition of ∂:

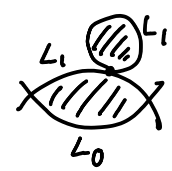
- transversality is achieved for simple maps by picking generic J.
  For multiply covered maps (or configs with complicated bubbling), need various tricks... (domain-dependent J's, multivalued perturbations, virtual cycles, ...)
- orientation on moduli space: need auxiliary data, & top. assumption on L_i.
  Namely: if equip L_i with spin structures (ie. double cover of frame bundle) then we get an orientation on moduli spaces.
- Compactness: again relies on Gromov's compactness theorem (for a fixed energy bound)
  there are now 3 types of phenomena.

→ bubbling of spheres: $|du_n| \to \infty$ at interior point
limiting configuration looks like



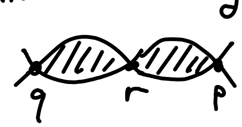
Treatment similar to case of closed curves / GW invariants
In good cases (if transversality can be achieved), configs. with sphere bubbles are a codim ≥ 2 subset of the compactified $\overline{\mathcal{M}}$

→ bubbling of discs: $|du_n| \to \infty$ at boundary point
limiting configuration looks like



This is a serious technical issue because, in transverse case, bubbled configurations have real $\mathrm{codim}_{\mathbb{R}}\, 1$, and contribute to $\partial \overline{\mathcal{M}}$.

→ breaking of strips: (or: energy escapes towards $s \to \pm\infty$
ie. reparametrizing $u_n(\cdot - \delta_n, \cdot)$ have $\neq$ limits)
(or, if using domain $= D^2 - \{\pm 1\}$, this is bubbling at ± 1).
Limiting configuration looks like



This is as in Morse theory, where a sequence of gradient flow lines can converge to a broken flow line

④ ★ How to prove $\partial^2 = 0$ assuming no bubbling:

consider $\mathcal{M}(p, q, \phi, J)/\mathbb{R}$ for $\phi \in \pi_2$ of index 2, J generic

This is expected to be a 1-dim'l mfld, which can be compactified by adding in broken trajectories $\coprod_{\substack{r \in L_0 \cap L_1 \\ \phi_1 \# \phi_2 = \phi}} \left(\mathcal{M}(p, r, \phi_1, J)/\mathbb{R}\right) \times \left(\mathcal{M}(r, q, \phi_2, J)/\mathbb{R}\right)$

(This is assuming no bubbling $\Rightarrow$ no other limiting scenarios!)

<u>Gluing thm</u> $\Rightarrow$ the resulting $\overline{\mathcal{M}(p, q, \phi, J)/\mathbb{R}}$ is a manifold w/ boundary.

Now: # ends of a compact 1-manifold, oriented (or counting mod 2), $= 0$.

Ends = contributions to coefficient of $T^{\omega(\phi)} q$ in $\partial^2(p)$.

★ Bubbling of discs is not just a technical issue to overcome, it's an actual obstruction to defining Floer homology