(1) <u>Recall:</u> quintic mirror family $\check{X}_\psi$, with LCSL degeneration as $z=(5\psi)^{-5}\to 0$

We studied Picard-Fuchs eq^{n} and found two solutions $\phi_0(z)=\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n$,

and $\phi_1(z)=\phi_0(z)\log z+\tilde{\phi}(z)$, $\tilde{\phi}(z)=5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)z^n$

Then we found the canonical coordinates: $\beta_0,\beta_1\in H_3(\check{X},\mathbb{Z})$ s.t. monodromy preserves β_0 & maps $\beta_1\mapsto\beta_1+\beta_0$

$\rightsquigarrow \int_{\beta_i}\check{\Omega}$ are linear combinations of ϕ_0 and ϕ_1

$\rightsquigarrow$ $w=\dfrac{\int_{\beta_1}\check{\Omega}}{\int_{\beta_0}\check{\Omega}}$, $\quad q=\exp(2\pi i w)=\underset{\uparrow}{c_2}\, z\exp\left(\dfrac{\tilde{\phi}(z)}{\phi_0(z)}\right)$

normalization constant (β_1 only def^{d} up to adding mult. of β_0).

- Yukawa coupling on $H^{2,1}(\check{X})$:

let $W_k=\int_{\check{X}_z}\check{\Omega}(z)\wedge\frac{d^k}{dz^k}\check{\Omega}(z)$ (still same family of $\check{\Omega}$!).

Rewrite Picard Fuchs in form $\frac{d^4}{dz^4}[\check{\Omega}]+\sum_{k=0}^{3}C_k(z)\frac{d^k}{dz^k}[\check{\Omega}]=0$

$\rightsquigarrow$ then $W_4+\sum_{k=0}^{3}C_kW_k=0$

But Griffiths transversality $\Rightarrow$ $\check{\Omega}$ & 1st, 2nd derivatives have no $(0,3)$ component $\Rightarrow$ $W_0=W_1=W_2=0$.

Moreover:

$$0=\frac{d^2W_2}{dz^2}=\int\frac{d^2\check{\Omega}}{dz^2}\wedge\frac{d^2\check{\Omega}}{dz^2}+2\int\frac{d\check{\Omega}}{dz}\wedge\frac{d^3\check{\Omega}}{dz^3}+\int\check{\Omega}\wedge\frac{d^4\check{\Omega}}{dz^4}$$

$$=0+2\left(\frac{dW_3}{dz}-W_4\right)+W_4$$

So $W_4=2W_3'$, and get $W_3'+\frac{1}{2}C_3W_3=0$!

Look at coeff^{t} of $\frac{d^3}{dz^3}$ in Picard-Fuchs $\rightsquigarrow C_3(z)=\frac{6}{z}-\frac{2.5^5}{1-5^5z}$

$\Rightarrow(\log W_3)'=-\frac{3}{z}+\frac{5^5}{1-5^5z}$. Integrating we get

②

$$W_3(z) = \frac{c_1}{(2\pi i)^3 z^3 (5^5 z - 1)}.$$ This is almost $\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle$.

Still need to <u>normalize</u> (want $\langle \dots \rangle$ rel. to $\frac{\check{\Omega}}{\int_{\beta_0} \check{\Omega}}$ not $\check{\Omega}$)

and switch to <u>canonical coordinates</u> (want $\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle$ not $\frac{\partial}{\partial z}$).

- normalization: scaling $\check{\Omega}$ by $f(z)$ changes $\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \int_X \check{\Omega} \wedge \frac{\partial^3}{\partial z^3} \check{\Omega}$ by $f(z)^2$
(no derivatives of f come up because $\check{\Omega} \wedge \frac{\partial^i}{\partial z^i} \check{\Omega} \equiv 0$ for $i<3$)

In our case, want to scale by $\frac{1}{\int_{\beta_0} \check{\Omega}} = \frac{\text{const}}{\phi_0(z)}$

$$\Rightarrow \langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \frac{c_1}{(2\pi i)^3 z^3 (5^5 z - 1)\, \phi_0(z)^2}$$

- switching coordinates: $\frac{\partial}{\partial w} = \left(\frac{dw}{dz}\right)^{-1} \frac{\partial}{\partial z} \Rightarrow \langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = \frac{c_1}{(5^5 z - 1)\phi_0(z)^2 \delta(z)^3}$

where $\delta(z) = 2\pi i\, z \frac{dw}{dz} = z \frac{d \log q}{dz} = 1 + z \frac{d}{dz}\left(\frac{\tilde{\phi}(z)}{\phi_0(z)}\right)$

Finally we want to expand this as a power series in q.
Since $dq/dz = q \frac{d\log q}{dz} = \frac{q}{z} \delta(z) = c_2\, \delta(z) \exp(\tilde{\phi}/\phi_0)$, we have

$$\frac{d^j}{dq^j}\left(\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle\right) = \left(\frac{1}{c_2 \delta(z) \exp(\tilde{\phi}/\phi_0)} \frac{d}{dz}\right)^j \left(\frac{c_1}{(5^5 z - 1)\phi_0(z)^2 \delta(z)^3}\right)$$

$\rightarrow$ calculate expansion in q by evaluating these at $z=0$ (from expansion of $\phi_0(z)$, $\tilde{\phi}(z)$)

get: $$\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = -c_1 - 575 \frac{c_1}{c_2} q - \frac{1950750}{2} \frac{c_1}{c_2^2} q^2 - \frac{10277490000}{6} \frac{c_1}{c_2^3} q^3 - \frac{7448604862500 0}{24} \frac{c_1}{c_2^4} q^4 + \dots$$

- Now recall: under mirror symmetry, expect $\exists$ basis $\{e\}$ of $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$
(i.e. $e = PD(\text{hyperplane})$) s.t., writing $[B + i\omega] = t e$, $q = \exp(2\pi i t) = e^{2\pi i \int_{\text{line}} B + i\omega}$

Then the mirror map is $q \longleftrightarrow q$

I.e. $w = \frac{1}{2\pi i} \log q \longleftrightarrow t$,

$\frac{\partial}{\partial w} \longleftrightarrow \frac{\partial}{\partial t}$ $\leftarrow$ <u>this</u> is the identification we want between $H^{2,1}(\check{X}) \simeq H^{1,1}(X)$

(Rather: $\frac{\partial}{\partial t} \in T\mathcal{M}_{\text{käh}} \longleftrightarrow \frac{\partial}{\partial t}[B + i\omega] = e \in H^{1,1}$).

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Now on sympl. side the Yukawa coupling was

$$\langle \tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial t} \rangle = \langle e,e,e \rangle = \int_X e \wedge e \wedge e + \underbrace{\sum_{d>0}}_{\text{degree}} \underbrace{\langle e,e,e \rangle_{0,d}}_{\text{GW inv't}} \, q^d$$

where $\langle e,e,e\rangle_{0,d} = \underbrace{\left(\int_d e\right)\left(\int_d e\right)\left(\int_d e\right)}_{= d^3} N_d$, $N_d = \sum_{d = kd'} \frac{n_{d'}}{k^3}$

$$\Rightarrow \langle e,e,e \rangle = 5 + \sum_{d=1}^{\infty} d^3 N_d \, q^d = 5 + \sum_{d=1}^{\infty} d^3 n_d \frac{q^d}{1-q^d}$$

$$= 5 + n_1 q + 8\left(n_2 + \frac{n_1}{8}\right) q^2 + 27\left(n_3 + \frac{n_1}{27}\right) q^3 + 64\left(n_4 + \frac{n_2}{8} + \frac{n_1}{64}\right) q^4 + \dots$$

Match with above $\Rightarrow$
- $c_1 = -5$
- $n_1 = \dfrac{575 \cdot 5}{c_2} = \dfrac{2875}{c_2}$

In fact, classical alg. geom. $\Rightarrow n_1 = 2875$, so $c_2 = 1$.

Then get the others:
- $n_2 = 609250$ (had been obtained by S. Katz 1986)
- $n_3 = 317206375$ (checked by Ellingsrud-Strømme 1990)
- $n_4 = 242467530000$

General verification (by pf of mirror symmetry for quintic in sense stated here) by Givental and Lian-Liu-Yau ~1996.

— more generally, for CY obtained as complete intersections in toric varieties.

We'll now switch to a different mathematical formulation of mirror symmetry, due to Kontsevich ~1994 : <u>homological mirror symmetry</u>

- On the symplectic side, we used to look at J-holomorphic spheres to get a "quantum" version of intersection pairing on $H^*(X)$, now we'll look at intersections of Lagrangian submanifolds, and "quantum" intersections theory involving J-holomorphic disks with boundary on Lagrangians
- On the complex side, we'll look at $\left\{ \begin{array}{l} \text{intersections of subvarieties} \\ \text{holomorphic maps / extensions of bundles / sheaves} \end{array} \right.$

④ We'll outline briefly the constructions of:

- the Fukaya (A_∞)-category (roughly: objects = Lagr. submflds
  morphisms = intersections
  alg. structure (diff^l, product, ...) = hol. disks)
- the cat. of coherent sheaves

HMS states the corresponding <u>derived categories</u> are equivalent
We'll try to understand a simple example $= T^2$

<u>Lagrangian Floer homology</u>:

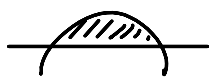
(M,ω) symplectic manifold $\supset L_0, L_1$ compact Lagrangian submanifolds

Assume $L_0 \pitchfork L_1 \implies L_0 \cap L_1$ finite set.

Recall Novikov ring $\Lambda = \{ \sum a_i T^{\lambda_i} \,/\, \lambda_i \to +\infty \}$

Floer complex $CF(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ free Λ-module gen^d by $L_0 \cap L_1$.

<u>Goal</u>: define a differential ∂ s.t. $HF(L_0,L_1) = H^*(CF, \partial)$ is invariant under Ham. isotopies ($\rightsquigarrow$ Arnold conj. on Lagr. intersections).
(namely HF gives obstruction to disjoining L_0, L_1).

In <u>topological</u> isotopies, pairs of $\cap$'s are cancelled along Whitney discs  ; in Hamiltonian isotopies, cancellations involve <u>holomorphic</u> Whitney discs.

<u>Look at</u>: $u : \mathbb{R} \times [0,1] \longrightarrow M$ equipped with J ω-compat. a.c.s.

$$
\text{s.t.}\quad (*) \begin{cases} \bullet\ \bar\partial_J u = 0, \text{ ie. } \dfrac{\partial u}{\partial s} + J \dfrac{\partial u}{\partial t} = 0 \\ \bullet\ u(s,0) \in L_0, \quad u(s,1) \in L_1 \\ \bullet\ \lim\limits_{s \to +\infty} u(s,t) = p, \quad \lim\limits_{s \to -\infty} u(s,t) = q \end{cases}
$$