* Schedule: TR 11-12:30 in 2-142
* Web page: math.mit.edu/~auroux/18.969/

Goal: overview of various aspects of mirror symmetry.

What is mirror symmetry? ... a phenomenon?
a conjecture?
a philosophy? ...

0. The physical origins: [& the only physics in this class !!]

- supersymmetric string theories:

propagation of a string in spacetime → surface Σ ("worldsheet")
→ 2D quantum field theory (on Σ), where some fields take values in manifolds (spacetime)

Require this theory to be supersymmetric (various operators act on the space of states preserving all observables) and conformal (invt under conformal changes of metric on Σ).

- Most common instance of such a theory: "nonlinear σ-model" determined by data := a Calabi-Yau manifold
  - (X,J) smooth complex manifold (ideally: compact, $\dim_{\mathbb{C}} 3$, maybe even $b_1=0$)
  - $K_X := \Lambda^n T^*X \underset{\text{holom.}}{\simeq} \mathcal{O}_X$, so $\exists\, \Omega \in H^0(X,K_X) = \Omega^{n,0}(X)$ nonvanishing holomorphic volume form
  - $\omega^{\mathbb{C}} = B + i\omega \in \Omega^{1,1}(X)$ complexified Kähler form: (closed, $\mathrm{Im}\,\omega^{\mathbb{C}} = \omega$ nondegenerate).

Deformations of the theory: $\begin{cases} \to \text{deformations of } J: \ H^1(X,TX) \\ \to \text{deformations of } \omega^{\mathbb{C}}: \ H^{1,1}(X) = H^1(X,\Omega^1_X) \end{cases}$

Dolbeault or sheaf cohomology

In fact, $\exists$ supersymmetry operators $Q, \overline{Q}$ ("charges") whose simult. eigenspaces are $H^q(\Lambda^p TX)$ and $H^q(X,\Omega^p_X)$

- Moreover, upon changing labelling of supersymmetry operators $((Q,\bar{Q}) \leftrightarrow (-Q,\bar{Q}))$ should get a new sigma-model in which eigenspaces are exchanged... can't do this on the same manifold, but hope $\exists (X^\vee, J^\vee, \omega^\vee)$ Calabi-Yan st.

$$H^q(X, \Lambda^p TX) \simeq H^q(X^\vee, \Omega^p_{X^\vee}) \quad \& \text{ vice versa}$$

+ st. superconformal field theories are equivalent

Then say $(X, J, \omega^{\mathbb{C}})$ & $(X^\vee, J^\vee, \omega^{\mathbb{C}\vee})$ are <u>mirrors</u>.

- There's two "topologically twisted" variants, "A-model" and "B-model"

A-model only depends on $\omega^{\mathbb{C}}$, not J — <u>symplectic geometry</u>

B-model only depends on J, not $\omega^{\mathbb{C}}$ — <u>Complex geometry</u>

Then A-model on $(X, \omega^{\mathbb{C}})$ is equivalent to B-model on $(X^\vee, J^\vee)$

B-model (X, J) A-model $(X^\vee, \omega^{\mathbb{C}\vee})$.

- For this to become a mathematical statement, need some consequences of equivalence of SCFTs that can be defined in mathematical terms!

<u>Answer:</u> "<u>correlation functions</u>" (= expectation values for observables ....) will have a mathematical formulation

(SCFTs themselves involve integrals over $\mathrm{Map}(\Sigma, X)$??)

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1. <u>Hodge theory, quantum cohomology</u>:

- First thing we expect of mirror CYs: $H^q(X, \Lambda^p T_X) \simeq H^q(X^\vee, \Omega^p_{X^\vee})$

CY assumption $\Rightarrow$ $\Lambda^p T_X \xrightarrow{\sim} \Omega^{n-p}_X$

$v_1 \wedge \dots \wedge v_p \longmapsto \iota_{v_1} \cdots \iota_{v_p} \Omega$

so we're saying: $\underline{H^{n-p,q}(X) \simeq H^{p,q}(X^\vee)}$

This is rather unusual: in dim. 3, it implies:

$$H^3(X, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}(X) \cong H^{0,0} \oplus H^{1,1} \oplus H^{2,2} \oplus H^{3,3}(X^\vee)!$$

- Moreover, correlation f^ns give extra structures on these spaces.

"Yukawa couplings" on $H^{1,1}$ and on $H^1(X,TX)$:

On $H^{1,1}(X)$: $\langle \omega_1, \omega_2, \omega_3 \rangle := \int_X \omega_1 \wedge \omega_2 \wedge \omega_3$

$$+ \sum_{\substack{\beta \in H_2(X,\mathbb{Z}) \\ \beta \neq 0}} n_\beta \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \; \frac{e^{2\pi i \int_\beta \omega}}{1 - e^{2\pi i \int_\beta \omega}}$$

where n_β = "number of genus 0 complex curves in X representing the homology class β"

what does that mean? (e.g., can be ∞ ??)

In fact: Gromov-Witten invariant

We'll also see.. even though it seems to involve J, actually n_β only depends on ω!

On $H^1(X,T_X) \simeq H^{2,1}(X)$: $\langle \theta_1, \theta_2, \theta_3 \rangle := \int_X \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega)$

where
$$\underset{\substack{\theta_1 \otimes \theta_2 \otimes \theta_3 \\ (0,1)\text{-forms in } T_X}}{H^1(X,T_X)^{\otimes 3}} \otimes \underset{\substack{\Omega \\ (3,0)\text{-form}}}{H^0(X,\Omega^3_X)} \to \underset{(3,3)\text{-form w/ values in } \Lambda^3 TX}{H^3(X, \Lambda^3 T_X \otimes \Omega^3_X)} = H^3(X,\mathcal{O}_X) = \underset{\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega}{H^{0,3}(X)}$$

or, in fact, thinking of θ_i as deform^ns of cx. structure J

→ Ω changes as we deform J ⇝ $\langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \Omega \wedge \nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega$

↑ Gauss-Manin connection on $H^*(X,\mathbb{C})$

Need of course to assume Ω normalized for this to be ok.

Mirror symmetry prediction: if X & $X^\vee$ are mirror, then

$\langle \cdot,\cdot,\cdot \rangle$ on $H^{1,1}(X) \simeq H^1(X^\vee, TX^\vee)$, $\langle \cdot,\cdot,\cdot \rangle$

match under a certain change of coordinates.

More precisely: relation between $\omega^{\mathbb{C}}$ on X and $J^\vee$ on $X^\vee$ should yield a local diffeo between moduli spaces $\mathcal{M}_{sympl}(X)$, $\mathcal{M}_{cx}(X^\vee)$ (the "mirror map"). At the level of tangent spaces, mirror map induces $T_{\omega^{\mathbb{C}}}\mathcal{M}_{sympl}(X) = H^{1,1}(X) \simeq H^1(X^\vee, TX^\vee) = T_{J^\vee}\mathcal{M}_{cx}(X^\vee)$

Then $\langle \cdot,\cdot,\cdot \rangle$ match under this isomorphism.

- First concrete math prediction made using mirror symmetry:

  (Candelas - de la Ossa - Green - Parkes, 1991)

  X = quintic Calabi-Yau 3-fold := degree 5 hypersurface in $\mathbb{CP}^4$

  $\rightsquigarrow$ predicted n_d = # deg. d rational curves in X

  e.g.: $n_1 = 2875$ lines on the quintic (already known)

  $n_2 = 609250$

  values for large d were previously unknown

Method:

- $X^\vee$ = (resolution of sings. of quotient of a quintic by $(\mathbb{Z}/5)^3$).
- calculated Yukawa coupling on $H^1(X^\vee, TX^\vee)$
- Identified mirror map
- deduced Yukawa coupling on $H^{1,1}(X)$ & hence n_d's.

- Goal for 1st month of course:
  - Calabi-Yau mflds, complex deform.^{n} theory
  - review of Hodge theory
  - pseudohol. curves, GW invariants
  - understanding the statement of mirror symmetry
  - example: the quintic 3-fold. [skipping lots of details].

Then we'll move on to more recent developments & other interpretations of mirror symmetry.

2. Homological mirror symmetry (Kontsevich '94)

Open string propagation $\rightsquigarrow$ worldsheet is a surface w/ boundary
Constraints on values of fields at boundary = "D-branes"
field theory axioms $\Rightarrow$ D-branes form a category

In A-model, branes are Lagrangian submanifolds + flat bundles
B-model, branes are complex analytic submanifolds + holom. bundles

Kontsevich gave a mathematically precise formulation:

HMS Conj: if $(X, J, \omega^{\mathbb{C}})$ and $(X^\vee, J^\vee, \omega^{\mathbb{C}\vee})$ are mirror
then
$$D^b\mathrm{Fuk}(X, \omega^{\mathbb{C}}) \simeq D^b\mathrm{Coh}(X^\vee, J^\vee)$$
$$D^b\mathrm{Coh}(X, J) \simeq D^b\mathrm{Fuk}(X^\vee, \omega^{\mathbb{C}\vee})$$

equivalence between triangulated categories

→ Fukaya cat: objects = Lagrangian submanifolds + ----
morphisms = intersection theory
algebraic structures from Floer homology

→ coherent sheaves: objects = coherent sheaves
morphisms = hom's & ext's of sheaves
(think of as intersection theory!)

NB: ex. of coherent sheaf includes: ($Y \subset X$ subvariety + $E \to Y$ holom. vect. bundle)

derived category: algebraic enlargement
(complexes of objects, up to homotopy, ...)
→ triangulated category, ie. $\exists$ notion of exact triangles
$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] & & C \arrow[ll, "{[1]}"] \end{tikzcd}$$

This puts a healthy dose of homological algebra into mirror symmetry... which we'll largely ignore to focus on geometry.

Relation to previous: Hochschild cohomology
$$QH^*(X) \longrightarrow HH^*(D^b\mathrm{Fuk}(X))$$
$$H^*(X, \Lambda TX) \underset{HKR}{\xrightarrow{\simeq}} HH^*(D^b\mathrm{Coh}(X))$$

Learn about: - Lagrangian submanifolds, Floer homology, a simplified, naive version of Fukaya category
- coherent sheaves, derived category
- example: the elliptic curve (after Polishchuk–Zaslow)

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3. Strominger-Yau-Zaslow conjecture: (1996)

What is it geometrically that makes manifolds mirrors of each other?

SYZ conj: (rough version) | $X, X^\vee$ mirrors $\Rightarrow$ they carry mutually dual special Lagrangian torus fibrations

$$\begin{tikzcd} T^n \arrow[r] & X \arrow[dr] & & \check{X} \arrow[dl] & \check{T}^n \arrow[l] \\ & & B & & \end{tikzcd}$$

- $L^n \subset (X, \omega, J, \Omega)$ is special Lag. if $\begin{cases} \omega_{|L} = 0 \\ \operatorname{Im} \Omega_{|L} = 0 \end{cases}$

(then $\operatorname{Re} \Omega_{|L} = \mathrm{vol}_{g|L}$ calibration)

- dual torus: $\check{T} = \operatorname{Hom}(\pi_1 T, U(1))$ flat $U(1)$-connections on T

Motivation from HMS: $p \in \check{X}$ point $\rightsquigarrow \mathcal{O}_p \in D^b\mathrm{Coh}(\check{X}) \leftrightarrow L_p \in D^b\mathrm{Fuk}(X)$?

$\underset{\text{gr. vect. space}}{\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \simeq H^*(T^n, \mathbb{C})}$ so L_p = Lag. torus + $U(1)$-connection?

The conjecture doesn't quite hold as stated... $\begin{cases} \to \text{near large complex limit} \\ \to \text{instanton corrections} \end{cases}$

We'll see: - special Lagrangians and their deformations
- Lag. fibrations, affine geometry
- Ex: elliptic curve
- instanton corrections ... more examples (K3, ...)

Finally: 4. extension to non-Calabi-Yau setting: Landau-Ginzburg models and their geometry; mirror symmetry for Fano varieties; Example: toric varieties.