PATH SPACES, CONTINUOUS TENSOR PRODUCTS, AND E_0-SEMIGROUPS

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Abstract. We classify all continuous tensor product systems of Hilbert spaces which are "infinitely divisible" in the sense that they have an associated logarithmic structure. These results are applied to the theory of E_0-semigroups to deduce that every E_0-semigroup possessing sufficiently many "decomposable" operators must be cocycle conjugate to a CCR flow.

A *path space* is an abstraction of the set of paths in a topological space, on which there is given an associative rule of concatenation. A *metric path space* is a pair (P, g) consisting of a path space P and a function $g : P^2 \to \mathbb{C}$ which behaves as if it were the logarithm of a multiplicative inner product. The logarithmic structures associated with infinitely divisible product systems are such objects. The preceding results are based on a classification of metric path spaces.

Contents

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Introduction. Let $\alpha = \{\alpha_t : t > 0\}$ be an E_0-semigroup. That is, each α_t is a normal unit-preserving $*$-endomorphism of $\mathcal{B}(H)$ such that $\alpha_s \circ \alpha_t = \alpha_{s+t}$, and which is continuous in the sense that for all $A \in \mathcal{B}(H)$, $\xi, \eta \in H$, $\langle \alpha_t(A)\xi, \eta\rangle$ is continuous in t.

A *unit* for α is a strongly continuous semigroup $U = \{U(t) : t \geq 0\}$ of bounded operators on H satisfying

$$\alpha_t(A)U(t) = U(t)A,$$

for every $t \geq 0$, $A \in \mathcal{B}(H)$. α is called *completely spatial* if there is a $t > 0$ such that H is spanned by the ranges of all operator products of the form

$$U_1(t_1)U_2(t_2)\dots U_n(t_n)$$

where $U_1, U_2, \dots, U_n$ are units, $t_1, t_2, \dots, t_n$ are nonnegative reals summing to t, and n is an arbitrary positive integer. Completely spatial E_0-semigroups are those which have "sufficiently many" units.

In [2, section 7], completely spatial E_0-semigroups are shown to be completely classified up to cocycle conjugacy by their numerical index. This is established at the level of continuous tensor product systems. In somewhat more detail, a *product system* is a measurable family of separable Hilbert spaces $E = \{E_t : t > 0\}$ which is endowed with an associative multiplication that "acts like tensoring" in the sense that for any choices $x, x' \in E_s$, $y, y' \in E_t$, the products xy, $x'y'$ both belong to E_{s+t} and we have

$$\langle xy, x'y'\rangle = \langle x, x'\rangle \langle y, y'\rangle, \qquad \text{and}$$
$$E_{s+t} = \overline{span} E_s E_t.$$

In addition, there is a third key axiom which plays the role of local triviality for Hermitian vector bundles. The precise definition of product systems can be found in [2, Definition 1.4]. The intuition is that E_t resembles a continuous tensor product

$$E_t = \bigotimes_{0<s<t} H_s, \qquad H_s = H \tag{I.1}$$

of copies of a single Hilbert space H. However, this heuristic picture of continuous tensor products cannot be pushed too far. While the formula (I.1) can be made precise for certain standard examples, it is also known that there are many product systems for which the "germ" H fails to exist.

Every E_0-semigroup α determines a product system E_α. The t^{th} Hilbert space $E_\alpha(t)$ is the linear space of operators

$$E_\alpha(t) = \{T \in \mathcal{B}(H) : \alpha_t(A)T = TA, \forall A \in \mathcal{B}(H)\},$$

with inner product defined by

$$\langle S, T\rangle \mathbf{1} = T^*S, \qquad S, T \in E_\alpha(t),$$

and with multiplication given by ordinary operator multiplication. It is known that two E_0-semigroups are cocycle conjugate iff their product systems E_α, E_β

are isomorphic [2, Corollary of Theorem 3.18]; moreover, every abstract product system is associated with some E_0-semigroup [5, Corollary 5.17].

These remarks show that, up to cocycle conjugacy, the theory of E_0-semigroups is equivalent to the theory of continuous tensor product systems. Consequently, a central component of our approach to E_0-semigroups has been to develop the theory of product systems. The classification of E_0-semigroups described above was obtained by classifying product systems which possess a property that corresponds to complete spatiality of their product systems (this was called *divisibility* in [2]).

In this paper, we extend that result so as to include certain product systems which do not appear to contain any units *a priori*, but which do contain sufficiently many vectors that resemble "elementary tensors". Such product systems are called *decomposable*. The corresponding property of E_0-semigroups is described as follows. Fix $t > 0$. An operator $T \in E_\alpha(t)$ is called *decomposable* if, for every $0 < s < t$ there are operators $A \in E_\alpha(s)$, $B \in E_\alpha(t-s)$ such that $T = AB$. Let $\mathcal{D}(t)$ denote the set of all decomposable operators in $E_\alpha(t)$. It is easy to see that if H is spanned by vectors of the form

$$\mathcal{D}(t_0)H = \{T\xi : T \in \mathcal{D}(t_0), \xi \in H\}$$

for some particular $t_0 > 0$, then we have

$$H = [\mathcal{D}(t)H] \tag{I.2}$$

for every $t > 0$. α is called *decomposable* if (I.2) is valid for some $t > 0$. Notice that if $U_1, U_2, \dots, U_n$ are units for α and $t_1, t_2, \dots, t_n$ are positive numbers summing to t, then every operator product of the form

$$U_1(t_1)U_2(t_2)\dots U_n(t_n)$$

is a decomposable operator in $E_\alpha(t)$ because each U_j is a semigroup. Thus, every completely spatial E_0-semigroup is decomposable.

On the other hand, decomposable E_0-semigroups are not required to contain any units *a priori*. The results of this paper imply that decomposable E_0-semigroups are completely spatial, and are therefore classified to cocycle conjugacy by their numerical index. In particular, decomposable E_0-semigroups necessarily have plenty of units.

Decomposability translates into an important property of abstract product systems, and we want to discuss this property and its role in structural issues. Let $E = \{E(t) : t > 0\}$ be an abstract product system and fix $t > 0$. A nonzero vector $x \in E(t)$ is called *decomposable* if for every $s \in (0, t)$ there are vectors $a \in E(s)$, $b \in E(t-s)$ such that $x = ab$. Let $D(t)$ be the set of all decomposable vectors in $E(t)$. $D(t)$ can be empty. But if it is large enough that it spans $E(t)$ for some particular t, then it is easy to show that

$$E(t) = \overline{span} D(t) \tag{I.3}$$

for every positive t. A product system satisfying (I.3) for every positive t is called *decomposable*. It is also easy to show that if E is isomorphic to the product system E_α of some E_0-semigroup α, then E is decomposable iff α is (Proposition 12.2).

If we think of the multiplication in a product system E as representing the tensor product operation, then the vectors in $D(t)$ are "elementary tensors". In the heuristic picture in which one thinks of $E(t)$ as a continuous tensor product

$$E(t) = \bigotimes_{0<s<t} H_s, \qquad H_s = H$$

of copies of a Hilbert space H, the decomposable vectors have a corresponding heuristic representation

$$x = \otimes_{0<s<t} x_s \tag{I.4}$$

where x_s is a nonzero vector in H for every $s \in (0,t)$. Since finite tensor products of Hilbert spaces of the form

$$\underbrace{H \otimes H \otimes \cdots \otimes H}_{n \text{ times}}$$

are always spanned by vectors of the form

$$x = x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

with $x_i \in H$, decomposability appears as a natural property of product systems. However, notice that Theorem 11.1 below implies that only the simplest product systems are decomposable.

Product systems occur naturally in probability theory. For example, any random distribution over the interval $(0,\infty)$ having stationary independent increments gives rise to a product system (see [2, pp 14–16] for an example based on Poisson processes). Some of these examples (such those arising from Poisson processes) do not appear to have "enough" units to be divisible in the sense of [2]. Nevertheless, they do. They do because in constructing these examples, one essentially constructs the sets $D(t)$, $t>0$, and then defines the product system E by setting

$$E(t) = \overline{span} D(t), \qquad t > 0.$$

Such a product system is clearly decomposable and thus, by Theorem 11.1 below, is simply one of the standard ones.

Our approach to decomposable product systems is based on the concept of a *metric path space*. Roughly speaking, a metric path space is a structure which behaves as if it were the "logarithm" of a product system. Path spaces are very general objects, and we give a variety of examples. The most natural constructions of product systems all begin with a metric path space of some kind...an example of this being the construction starting with a Poisson process as above. We give an essentially complete classification of metric path spaces in part I. In part II, we show that every decomposable product system is associated with an essentially unique metric path space, and in part III these results are applied to the classification of product systems and E_0-semigroups.

We still do not know a "natural" construction of a product system without any units, or even one with some units but not one of the standard ones. Such product systems are known to exist by work of Powers [25, 27] together with the results of [2]. Powers' constructions are very indirect. We believe that there should be a natural way of constructing such product systems, and we offer that as a basic unsolved problem. The fact that we do not yet know how to solve it shows how poorly understood continuous tensor products are today.

It is appropriate to discuss the relation of our results to the work of Araki and Woods on continuous tensor products [1]. Araki and Woods were interested in generalizing results of von Neumann on infinite tensor products of Hilbert spaces [21,22] to the case of continuous tensor products. Their precise formulation of a continuous tensor product of Hilbert spaces is expressed in terms of a nonatomic Boolean algebra of type I factors. Our formulation of continuous tensor products is in terms of the product systems defined above. While both concepts seek to make precise the intuitive idea of continuous tensor products of Hilbert spaces, they are formulated in substantially different and mathematically inequivalent ways.

For example, the Hilbert spaces of [1] are parameterized by the elements of a σ-algebra of sets, whereas the fiber spaces of product systems are parameterized by positive real numbers. If one starts with an abstract product system E then there is no underlying Hilbert space in view. Even if E did act naturally on some Hilbert space (as it does when it is defined in terms of an E_0-semigroup), one still cannot use it to define a Boolean algebra of type I factors. It is true that one can associate a type I factor with subintervals of $(0, \infty)$, but one does not know how to extend that set function to arbitrary Borel sets in $(0, \infty)$...certainly not so that one can be sure the extended set function will take values in the set of type I factors.

Thus, product systems are more general than the structures of Araki-Woods. But even that assertion is not exactly true. We also do not know how to start with a complete Boolean algebra of type I factors over the interval $(0, \infty)$ and use that structure to define a product system. In order to do that, one needs a bit more symmetry with respect to positive translations. Still, I am comfortable with the philosophical position that product systems are generalizations of complete Boolean algebras of type I factors over the interval $(0, \infty)$.

In more pragmatic terms, we have been unable to apply the work of Araki-Woods to the theory of product systems or to the theory of E_0-semigroups, even though there are apparent similarities in certain results. For example, Theorem 11.1 below is similar to Theorem 6.1 of [1]. Notice that the hypothesis of the latter, namely

$$\inf_B d(\Psi; B) > 0,$$

corresponds to our hypothesis of decomposability because of Lemma 6.3 of [1]. In fact, in establishing the classification results of [2] as well as those of the present paper, we have had to start from scratch...basing our proofs on different principles than the proofs of corresponding results of [1]. Nevertheless, we certainly acknowledge the considerable influence of [1] in guiding our thinking about continuous tensor products.

Part I: Path Spaces

1. Definitions and examples.

A *path space* is a family of sets over the positive real axis

$$\pi : P \to (0, \infty)$$

which is endowed with an associative multiplication $(x, y) \in P \times P \mapsto xy \in P$ satisfying the following two axioms. Letting $P(t) = \pi^{-1}(t)$ be the fiber over t we require that for every $s, t > 0$,

$$P(s) \cdot P(t) = P(s + t) \tag{1.1.1}$$

and that for all x_1, $x_2 \in P(s)$, y_1, $y_2 \in P(t)$

$$x_1 y_1 = x_2 y_2 \implies x_1 = x_2 \text{ and } y_1 = y_2. \tag{1.1.2}$$

Together, the conditions (1.1) assert that the projection π should obey $\pi(xy) = \pi(x) + \pi(y)$, and that for fixed $0 < s < t$, each element $z \in P(t)$ factors uniquely into a product $z = xy$, with $x \in P(s)$ and $y \in P(t - s)$.

Path spaces are generalizations of continuous cartesian products, whereby one starts with a basic set X and defines fiber spaces $P(t)$ by

$$P(t) = X^{(0,t]}. \tag{1.2.1}$$

P is defined as the total set $\pi : P = \{(t, f) : t > 0, f \in P(t)\} \to (0, \infty)$ with its natural projection $\pi(t, f) = t$. The multiplication in P is defined by

$$(s, f) \cdot (t, g) = (s + t, h)$$

where h is the concatenation

$$h(x) = \begin{cases} f(x), & 0 < x \leq s \\ g(x - s), & s < x \leq s + t. \end{cases} \tag{1.2.2}$$

Remarks. The terminology *path space* suggests that one should think of the elements of P as representing paths in some space. Notice however that if X is a topological space in the preceding example (1.2), the functions in any particular $P(t)$ are not necessarily continuous, nor even measurable. In fact, the examples (1.2) are pathological. We give some more typical examples below. But we emphasize that in general, path spaces are not required to support an additional topological structure, nor even a Borel structure. We will find that, nevertheless, the intrinsic structure of path spaces allows one to define a notion of measurability (see Definition 2.2) and even continuity (see section 8) for the functions defined on them.

Let $\pi : P \to (0, \infty)$ be a path space and let $x, y \in P$. We will say that x is a left divisor of y is there is an element $z \in P$ with $y = xz$. The element z is necessarily unique and, if $x \in P(s)$ and $y \in P(t)$ then we must have $0 < s < t$ and $z \in P(t - s)$. Fix $T > 0$. A section

$$t \in (0, T] \mapsto x(t) \in P(t)$$

is said to be *left-coherent* if $x(s)$ is a left divisor of $x(t)$ whenever $s < t$. Notice that such a section is uniquely determined by its last element $x(T) \in P(T)$ because of the unique factorization property. Thus there is a bijective correspondence between elements of $P(T)$ and left-coherent sections $t \in (0, T] \mapsto x(t) \in P(t)$.

There is another description of left coherent sections $t \in (0, T] \mapsto P(t)$ that we will find useful. Given such a section we can define a family of elements $\{x(s, t) \in P(t-s) : 0 \le s < t \le T\}$ by

$$x(t) = x(s)x(s, t), \qquad \text{for } 0 < s < t \le T,$$

and where $x(0, t)$ is defined as $x(t)$. This family satisfies the consistency equation

$$0 \le r < s < t \le T \implies x(r, t) = x(r, s)x(s, t). \tag{1.3}$$

There is an obvious modification of these considerations for fully defined sections $t \in (0, \infty) \mapsto x(t) \in P(t)$, and in this case we obtain a family $\{x(s, t) : 0 \le s < t < \infty\}$ satisfying (1.3) for unrestricted $0 \le r < s < t$.

Definition 1.4. *Let I be an interval of the form $I = [0, T]$ with $0 < T < \infty$ or $I = [0, \infty)$. A family of elements $\{x(s, t) \in P(t-s) : s, t \in I, s < t\}$ is called a propagator on I if it satisfies the equation (1.3) for every $r, s, t \in I$ such that $r < s < t$.*

Operator-valued functions satisfying equation (1.3) are naturally associated with solutions of time-dependent linear differential equations of the form

$$\frac{d}{dt}x(t) = x(t)a(t), \qquad 0 \le t \le T,$$

where $a(\cdot)$ is a given operator-valued function with invertible values [30, p. 282]. For our purposes, propagators are associated with left-coherent sections as above. Notice that the correspondence between propagators and left-coherent sections is also bijective since one can recover the section $\{x(t)\}$ from the propagator $\{x(s, t)\}$ by setting

$$x(t) = x(0, t).$$

Example 1.5. Let V be a finite dimensional vector space or a Banach space. For every $t > 0$, let $P(t)$ denote the space of all continuous functions

$$f : [0, t] \to V$$

satisfying $f(0) = 0$. For $f \in P(s)$, $g \in P(t)$ we define a concatenation $f * g \in P(s+t)$ by

$$f * g(\lambda) = \begin{cases} f(\lambda), & 0 \le \lambda < s \\ f(s) + g(\lambda - s), & s \le \lambda \le s + t. \end{cases}$$

Notice that $f * g$ executes the path f first, and then it executes g. If we assemble these spaces into a family

$$P = \{(t, f) : t > 0, f \in P(t)\}$$

with projection $\pi(t, f) = t$ and multiplication

$$(s, f) \cdot (t, g) = (s + t, f * g),$$

then we obtain a path space $\pi : P \to (0, \infty)$.

Example 1.6. One can define variations of example 1.5 with higher degrees of smoothness. For example, let $P(t)$ be the space of all continuously differentiable functions $f : [0,t] \to V$ satisfying $f(0) = f'(0) = 0$. Then with the concatenation rule in which for $f \in P(s)$ and $g \in P(t)$, $f * g$ is defined by

$$f * g(\lambda) = \begin{cases} f(\lambda), & 0 \leq \lambda < s \\ f(s) + (\lambda - s)f'(s) + g(\lambda - s), & s < \lambda \leq s + t \end{cases}$$

we obtain a path space structure on $P = \{(t, f) : t > 0, f \in P(t)\}$ by way of $\pi(t, f) = t$, $(s, f) \cdot (t, g) = (s + t, f * g)$.

Example 1.7. As we will see in section 4, the most important examples of path spaces arise as follows.

Let $\mathcal{C}$ be a separable Hilbert space, and consider the Hilbert space $L^2((0,\infty);\mathcal{C})$ of all square integrable vector valued functions $f : (0,\infty) \to \mathcal{C}$ with inner product

$$\langle f, g \rangle = \int_0^\infty \langle f(x), g(x) \rangle \, dx.$$

For every $t > 0$, let $\mathcal{P}_{\mathcal{C}}(t)$ denote the subspace of $L^2((0,\infty);\mathcal{C})$ consisting of all functions f satisfying $f(x) = 0$ almost everywhere for $x \geq t$. For $f \in \mathcal{P}_{\mathcal{C}}(s)$, $g \in \mathcal{P}_{\mathcal{C}}(t)$ we define the concatenation $f \boxplus g \in \mathcal{P}_{\mathcal{C}}(s + t)$ by

$$f \boxplus g(\lambda) = \begin{cases} f(\lambda), & 0 < \lambda < s \\ g(\lambda - s), & s \leq \lambda \leq s + t. \end{cases}$$

Let $\pi : \mathcal{P}_{\mathcal{C}} \to (0,\infty)$ be the total space defined by

$$\mathcal{P}_{\mathcal{C}} = \{(t, f) : t > 0, f \in \mathcal{P}_{\mathcal{C}}(t)\}, \tag{1.8}$$

where $\pi(t, f) = t$. With multiplication $(s, f) \cdot (t, g) = (s + t, f \boxplus g)$, $\pi : \mathcal{P}_{\mathcal{C}} \to (0,\infty)$ becomes a path space.

Needless to say, there are obvious variations of these examples which can be formulated in different function spaces.

Remarks. The preceding examples involve generalized paths in vector spaces. It is less obvious how one might concatenate paths in multiply connected spaces so as to obtain a path space structure. For example, suppose we are given a closed set $K \subseteq \mathbb{R}^2$ which represents an "obstruction" and one defines $P(t)$ to be the set of all continuous functions

$$f : [0, t] \to \mathbb{R}^2 \setminus K$$

satisfying $f(0) = 0$ (we assume, of course, that $0 \notin K$). One may not concatenate elements according to the rule of example 1.5 since that rule of concatenation will not necessarily provide a path $f * g$ that avoids K, even if f and g both avoid K. Nevertheless, there are many ways to define such concatenation operations. We offer the following digression for the interested reader, though we will make no further reference to this class of examples in this paper.

Example 1.9. Let K be a closed subset of $\mathbb{R}^n$, $n \geq 2$, having smooth boundary and which does not contain 0. Let

$$V : \mathbb{R}^n \setminus K \to \mathbb{R}$$

be any C^1 function which tends to $+\infty$ near the boundary of K in the sense that if x_n is any sequence for which the distance from x_n to K tends to 0, then $V(x_n) \to +\infty$. For example, one may take

$$V(x) = \sup_{y \in K} \frac{1}{|x - y|}, \qquad x \in \mathbb{R}^n \setminus K.$$

For such a V, the vector field

$$F(x) = -\nabla V(x)$$

becomes infinitely repulsive as x approaches K. If the vector $v = \nabla V(0)$ is not zero then we can replace V with

$$\tilde{V}(x) = V(x) - \sum_{k=1}^{n} v_k x_k$$

in order to achieve the normalization $\nabla V(0) = 0$.

Let $P(t)$ be the set of all continuously differentiable functions

$$f : [0, t] \to \mathbb{R}^n \setminus K$$

satisfying $f(0) = f'(0) = 0$. We can use the vector field F to construct a concatenation operation on the family $P = \{(t, f) : t > 0, f \in P(t)\}$ as follows. Starting with any continuous function $\phi : [0, t] \to \mathbb{R}^n$ with $\phi(0) = 0$, there is a corresponding element $f \in P(t)$ defined as the solution of the initial value problem

$$\begin{aligned} f'(s) &= \phi(s) - \nabla V(f(s)), \\ f(0) &= 0. \end{aligned}$$

Notice that $f'(0) = 0$ because of the normalization of V. There is no problem with the existence of global solutions of this differential equation because of the nature of V. Indeed, it is quite easy to show that this differential equation has a unique solution over the interval $0 \leq s < \infty$, and we obtain f by restricting the solution to the interval $[0, t]$. Conversely, every $f \in P(t)$ arises in this way from the "driving" function $\phi(s) = f'(s) + \nabla V(f(s))$, $0 \leq s \leq t$. Thus $f \leftrightarrow \phi$ defines a bijection of families of sets over $(0, \infty)$.

Thus we can transfer the concatenation of example 1.5 directly to define a concatenation $f, g \in P(s) \times P(t) \mapsto f * g \in P(s + t)$. Explicitly, given $f \in P(s)$, $g \in P(t)$, then $h = f * g$ is the element of $P(s + t)$ defined by the initial value problem

$$h'(\lambda) = \phi(\lambda) - \nabla V(h(\lambda))$$

$$h(0) = 0$$

on the interval $0 \le \lambda \le s+t$, where $\phi : [0, s+t] \to \mathbb{R}^n$ is the driving function

$$\phi(\lambda) = \begin{cases} f'(\lambda), & 0 \le \lambda < s \\ f'(s) + g'(\lambda - s) + \nabla V(f(s)) + \nabla V(g(\lambda - s)), & s \le \lambda \le s+t. \end{cases}$$

Notice that $f * g$ agrees with f on the interval $0 \le \lambda \le s$, but fails to agree with $f(s) + g(\lambda - s)$ on the interval $s \le \lambda \le s+t$.

Remark. There are many potentials that one could use, and thus there are infinitely many path space structures whose fiber spaces are of the form

$$P(t) = \{f : [0,t] \to \mathbb{R}^n \setminus K : f \in C^1, \quad f(0) = f'(0) = 0\},$$

$t > 0$. It is clear from the construction that all of these path spaces are isomorphic to the example 1.5. On the other hand, notice that if one is presented with just the path space structure of one of these examples, it will not be possible to write down the relevant isomorphism if one does not know the correct potential V (at least up to a constant).

2. Additive forms and multiplicative forms.

Given a set X and a positive definite function $B : X \times X \to \mathbb{C}$, one can construct a Hilbert space $H(X, B)$. If B and C are two positive definite functions

$$B : X \times X \to \mathbb{C}$$
$$C : Y \times Y \to \mathbb{C}$$

then the product $B \times C$ defines a positive definite function on the cartesian product $(X \times Y) \times (X \times Y)$ by way of

$$B \times C((x_1, y_1), (x_2, y_2)) = B(x_1, x_2)C(y_1, y_2),$$

and one has a natural isomorphism of Hilbert spaces

$$H(X \times Y, B \times C) = H(X, B) \otimes H(Y, C).$$

Thus, if one is given a path space $\pi : P \to (0, \infty)$ and a family of positive definite functions

$$B_t : P(t) \times P(t) \to \mathbb{C}, \qquad t > 0$$

which is multiplicative in an appropriate sense, then one would expect to obtain a continuous tensor product of Hilbert spaces...more precisely, a *product system* in the sense of [2].

The purpose of this section is to formulate these issues in the case where B_t is *infinitely divisible* in the sense that there is a family of conditionally positive definite functions

$$g_t : P(t) \times P(t) \to \mathbb{C} \qquad t > 0$$

such that $B_t = e^{g_t}$. Of course, multiplicative properties of B follow from appropriate additive properties of g. A classification of all product systems that can arise from this construction appears in Corollary 4.33.

We begin by reviewing a few basic facts and some terminology relating to conditionally positive definite functions [18]. Let X be a set and let $g : X \times X \to \mathbb{C}$ be a function. g is called conditionally positive definite if it is self-adjoint in the sense that $g(y,x) = \overline{g(x,y)}$ for all x, y, and is such that for every $x_1, x_2, \dots, x_n \in X$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$, we have

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j g(x_i, x_j) \geq 0.$$

If $p : X \times X \to \mathbb{C}$ is a positive definite function and $\psi : X \to \mathbb{C}$ is arbitrary then

$$g(x,y) = p(x,y) + \psi(x) + \overline{\psi(y)} \tag{2.1}$$

is conditionally positive definite. For g and p fixed, the function ψ satisfying (2.1) is not unique, but it is unique up to a perturbation of the form

$$\psi'(x) = \psi(x) + ic$$

where c is a real constant. Conversely, every conditionally positive definite function g can be decomposed into a sum of the form (2.1). For fixed g, the positive definite function appearing in (2.1) is not unique. If $g : X \times X \to \mathbb{C}$ is conditionally positive definite then

$$B(x,y) = e^{g(x,y)}$$

defines a positive definite function $B : X \times X \to \mathbb{C}$. The converse is false: there are self-adjoint functions $g : X \times X \to \mathbb{C}$ which are not conditionally positive definite whose exponentials $B = e^g$ are positive definite. However, if there is a sequence λ_n of positive numbers which tends to zero as $n \to \infty$ and

$$B_n(x,y) = e^{\lambda_n g(x,y)} \tag{2.2}$$

is positive definite for every $n = 1, 2, \dots$, then g is conditionally positive definite. Notice that if

$$B_n(x,y) = e^{\frac{1}{n} g(x,y)}$$

is positive definite for every $n = 1, 2, \dots$ then

$$B_1(x,y) = e^{g(x,y)} = B_n(x,y)^n$$

has a positive definite n^{th} root for every $n = 1, 2, \dots$; such a positive definite function B_1 is called *infinitely divisible.*

Let P be a path space, which will be fixed throughout this section. P^2 will denote the fiber product

$$P^2 = \{(t,x,y) : x, y \in P(t)\},$$

with projection $\pi(t,x,y) = t$ and fiber spaces $P^2(t) = P(t) \times P(t)$. A function $g : P^2 \to \mathbb{C}$ is called conditionally positive definite if for every $t > 0$, the restriction of g to $P(t) \times P(t)$ is conditionally positive definite.

Given such a function $g : P^2 \to \mathbb{C}$, we may construct a Hilbert space $H(t)$ for every $t > 0$. Briefly, letting $\mathbb{C}_0 P(t)$ denote the set of all functions $f : P(t) \to \mathbb{C}$ such that $f(x) = 0$ for all but finitely many $x \in P(t)$ and which satisfy $\sum_x f(x) = 0$, then $\mathbb{C}_0P(t)$ is a complex vector space and we may define a positive semidefinite sesquilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}_0P(t)$ by

$$\langle f_1, f_2\rangle = \sum_{x,y\in P(t)} f_1(x)\overline{f_2(y)}g(x,y).$$

After passing to the quotient of $\mathbb{C}_0P(t)$ by the subspace $\{f \in \mathbb{C}_0P(t) : \langle f, f\rangle = 0\}$ and completing the resulting inner product space, we obtain a Hilbert space $H(t)$. We will say that g is *separable* if $H(t)$ is a separable Hilbert space for every $t > 0$.

In spite of the fact that each fiber $P(t)$ is a lifeless set without additional structure, there is a useful notion of measurability for conditionally positive definite functions $g : P^2 \to \mathbb{C}$. Fix $0 < s < t$. For every element $y \in P(t)$ we may consider its associated propagator $\{y(\lambda,\mu) : 0 \le \lambda < \mu \le t\}$. Notice that for every $x \in P(s)$ and every λ in the interval $[0, t-s]$ we can form the complex number $g(x, y(\lambda, \lambda+s))$.

Definition 2.2. *g is called measurable if for every $0 < s < t < \infty$, every pair of elements $x_1, x_2 \in P(s)$ and every $y \in P(t)$,*

$$\lambda \in (0, t-s) \mapsto g(x_1, y(\lambda, \lambda+s)) - g(x_2, y(\lambda, \lambda+s))$$

defines a complex-valued Borel function.

Indeed, we will see that for the conditionally positive definite functions that are of primary interest, the functions appearing in Definition 2.2 are actually *continuous* (see Theorem 4.3). In view of the fact that we have imposed no structure on path spaces beyond that which follows from their rule of multiplication, this property appears noteworthy.

Finally, we introduce an appropriate notion of additivity for conditionally positive definite functions $g : P^2 \to \mathbb{C}$.

Definition 2.3. *g is called additive if, there is a function defined on the full cartesian product*

$$\psi : P \times P \to \mathbb{C}$$

such that for all $s, t > 0$, $x_1, x_2 \in P(s)$, $y_1, y_2 \in P(t)$,

$$g(x_1y_1, x_2y_2) - g(x_1, x_2) - g(y_1, y_2) = \psi(x_1, y_1) + \overline{\psi(x_2, y_2)}. \tag{2.4}$$

Remarks. Notice that the domain of ψ, namely $P \times P$, is larger than the domain of g, namely P^2. In fact, P^2 is the diagonal of $P \times P$:

$$P^2 = \{(p, q) \in P \times P : \pi(p) = \pi(q)\},$$

$\pi : P \to (o, \infty)$ being the natural projection.

The function ψ of Definition 2.3 is called the *defect* of g. The defect of g is not uniquely determined by equation (2.4), but if ψ_1 and ψ_2 both satisfy (2.4) then it is easy to see that there must be a function $c : (0,\infty) \times (0,\infty) \to \mathbb{R}$ such that $\psi_2(x, y) = \psi_1(x, y) + ic(s, t)$ for every $x \in P(s)$, $y \in P(t)$, $s, t > 0$. g is called *exact* if there is a function $\rho : P \to \mathbb{C}$ and a real-valued function $c : (0,\infty) \times (0,\infty) \to \mathbb{R}$ such that for every $s, t > 0$, $x \in P(s)$, $y \in P(t)$,

$$\psi(x, y) = \rho(xy) - \rho(x) - \rho(y) + ic(s, t). \tag{2.5}$$

Notice that when g is exact we can replace it with

$$g_0(x, y) = g(x, y) - \rho(x) - \overline{\rho(y)}$$

to obtain a new conditionally positive definite function $g_0 : P^2 \to \mathbb{C}$ which obeys the addition formula (2.4) with zero defect.

Definition 2.6. *Let P be a path space. An additive form is a function $g : P^2 \to \mathbb{C}$ which restricts to a conditionally positive definite function on $P(t) \times P(t)$ for every $t > 0$, and which is separable, measurable, and additive.*

A metric path space is a pair (P, g) consisting of a path space P and an additive form $g : P^2 \to \mathbb{C}$.

There are many natural examples of additive forms on path spaces. We give two simple ones here that are important for probability theory, and a third example which will be central to what follows.

For every $t > 0$ let $PC[0,t]$ denote the space of all piecewise continuous real-valued functions $f : [0,t] \to \mathbb{R}$ and let PC be the path space

$$\begin{aligned} PC &= \{(t, f) : t > 0, \quad f \in PC[0,t]\,\} \\ \pi(t, f) &= t \end{aligned}$$

with concatenation defined by $(s, f)(t, g) = (s + t, f * g)$ where

$$f * g(\lambda) = \begin{cases} f(\lambda), & 0 \le \lambda < s \\ g(\lambda - s), & s \le \lambda \le s + t. \end{cases}$$

Let c be a positive constant.

Example 2.7: Gaussian forms on PC. For $x_1, x_2 \in PC[0,t]$, put

$$g(x_1, x_2) = -c \int_0^t [x_1(\lambda) - x_2(\lambda)]^2 \, d\lambda.$$

Example 2.8: Poisson forms on PC. Let h be a second positive constant and for x_1, $x_2 \in PC[0,t]$ put

$$g(x_1, x_2) = c \int_0^t [e^{ih(x_1(\lambda) - x_2(\lambda))} - 1] \, d\lambda.$$

The forms g defined in examples 2.7 and 2.8 are essentially the covariance functions associated with random processes of the type indicated by their name. Note that in both cases the processes have stationary increments, and in fact 2.7 is the covariance function of Brownian motion. Notice too that the detailed structure of the path space used in these examples is not critical. For instance, if one replaces $PC[0,t]$ with the corresponding Skorohod space $D[0,t]$ and imitates what was done above, the new examples will share the essential features as those of 2.7 and 2.8.

More generally, with any continuous conditionally positive definite function of two real variables $\gamma : \mathbb{R}^2 \to \mathbb{C}$ we can associate an additive form g on PC by way of

$$g(x_1, x_2) = \int_0^t \gamma(x_1(\lambda), x_2(\lambda)) \, d\lambda,$$

for $x_1, x_2 \in PC[0,t]$.

Notice too that in all of the preceding examples the defect of g is zero. This will *not* be the case for additive forms that are associated with decomposable product systems as in in chapter II, and we will have to deal with additive forms having nonzero defects.

2.9: The standard examples. Let $\mathcal{C}$ be a separable Hilbert space and consider the path space $\mathcal{P}_\mathcal{C}$ of example 1.7. In this case the additive form $g : \mathcal{P}_\mathcal{C}^2 \to \mathbb{C}$ is simply the inner product inherited from $L^2((0,\infty);\mathcal{C})$,

$$g(f_1, f_2) = \int_0^t \langle f_1(\lambda), f_2(\lambda)\rangle \, d\lambda,$$

for $f_1,\ f_2 \in \mathcal{P}_\mathcal{C}(t)$.

If we replace $\mathcal{C}$ with a Hilbert space $\mathcal{C}'$ having the same dimension n as $\mathcal{C}$ then we obtain a new path space $\mathcal{P}_{\mathcal{C}'}$ and a new form

$$g' : \mathcal{P}_{\mathcal{C}'}^2 \to \mathbb{C}.$$

However, any unitary operator $W : \mathcal{C} \to \mathcal{C}'$ induces an obvious isomorphism of path space structures

$$\tilde{W} : \mathcal{P}_\mathcal{C} \to \mathcal{P}_{\mathcal{C}'}$$

by way of $\tilde{W} f(\lambda) = W f(\lambda)$ for $\lambda \in (0, t]$, $t > 0$. It follows that

$$g'(\tilde{W} f_1, \tilde{W} f_2) = g(f_1, f_2)$$

for $f_1,\ f_2 \in \mathcal{P}_\mathcal{C}(t)$, $t > 0$. We conclude that, up to isomorphism, the examples of 2.9 depend only on the dimension n. This sequence of metric path spaces will be denoted $(\mathcal{P}_n, g_n)$, $n = 1, 2, \dots, \infty$. It is convenient to include the degenerate pair $(\mathcal{P}_0, g_0)$ where $\mathcal{P}_0$ is the trivial path space $\mathcal{P}_0 = (0,\infty) \times \{0\}$ with multiplication $(s, 0)(t, 0) = (s + t, 0)$ and additive form $g_0(x, y) = 0$ for all x, y. Thus $(\mathcal{P}_n, g_n)$ is defined for every $n = 0, 1, 2, \dots, \infty$.

Suppose now that we are given a metric path space (P, g), and assume for the moment that g has *defect zero*. Then for every $t > 0$ we can define a positive definite function on $P(t)$ by

$$x, y \in P(t) \mapsto e^{g(x,y)}.$$

Let $E(t)$ be the Hilvert space obtained from this positive definite function. In more detail, there is a function $F_t : P(t) \to E(t)$ with the property that $E(t)$ is spanned by the range of F_t and

$$\langle F_t(x), F_t(y)\rangle = e^{g(x,y)}$$

for every $x, y \in P(t)$. It can be seen that the separability hypothesis on g implies that $E(t)$ is separable (in fact, the separability hypothesis implies that $E(t)$ can be identified with a subspace of the symmetric Fock space over a separable one-particle space as at the end of section 4).

Let us examine the consequences of the formula (2.4) with $\psi = 0$. Fixing $x_1, x_2 \in P(s)$ and $y_1, y_2 \in P(t)$, we have

$$e^{g(x_1 y_1, x_2 y_2)} = e^{g(x_1, x_2)} e^{g(y_1, y_2)}$$

and hence

$$\langle F_{s+t}(x_1 y_1), F_{s+t}(x_2 y_2)\rangle = \langle F_s(x_1), F_s(x_2)\rangle \langle F_t(y_1), F_t(y_2)\rangle$$

It follows that there is a unique bilinear map $(\xi, \eta) \in E(s) \times E(t) \mapsto \xi \cdot \eta \in E(s+t)$ which satisfies $F_s(x) \cdot F_t(y) = F_{s+t}(xy)$ for all $x \in P(s)$, $y \in P(t)$ and this extended mapping acts like tensoring in that

$$\langle \xi_1 \cdot \eta_1, \xi_2 \cdot \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle$$

for every $\xi_1, \xi_2 \in E(s)$, $\eta_1, \eta_2 \in E(t)$.

Thus we can define an associative operation in the total family of Hilbert spaces $p : E \to (0, \infty)$ defined by

$$E = \{(t, \xi) : t > 0, \quad \xi \in E(t)\}$$
$$p(t, \xi) = t$$

by way of $(s, \xi)(t, \eta) = (s + t, \xi \cdot \eta)$. This structure $p : E \to (0, \infty)$ has the main features of a product system [2]. However, since the total family E carries no natural Borel structure (because we are not given a Borel structure on the total space of P), the measurability axioms for product systems are meaningless here. For this reason, we will refer to $p : E \to (0, \infty)$ as the *product structure* associated with the metric path space (P, g).

The above construction required that g have defect zero. If g has nonzero defect but is exact, then one can modify this construction so as to obtain a product structure in this case as well (see section 4).

In general, Theorem 4.3 below implies that every additive form on a path space is exact. Moreover, we will find that (P, g) must be "essentially" isomorphic to one of the standard examples $(\mathcal{P}_n, g_n)$, $n = 0, 1, 2, \dots, \infty$ in such a way that the product structure associated with (P, g) is either the trivial one having one-dimensional spaces $E(t)$ or is isomorphic to one of the standard product systems of [2]. Thus, *every* metric path space gives rise to a product structure that is completely understood.

3. Exactness of cocycles. Let $\mathcal{C}$ be a separable Hilbert space and let L be the space of all measurable functions $f : (0, \infty) \to \mathcal{C}$ which are locally square integrable in the sense that

$$\int_0^T \|f(x)\|^2 \, dx < \infty$$

for every $T > 0$. The topology of L is defined by the sequence of seminorms

$$\|f\|_n = (\int_0^n \|f(x)\|^2 \, dx)^{1/2}$$

$n = 1, 2, \dots$ and

$$d(f, g) = \sum_1^\infty 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}$$

is a translation invariant metric on L with respect to which it becomes a separable Frèchet space.

L is the dual of the inductive limit of Hilbert spaces

$$L_0 = \bigcup L^2((0, t); \mathcal{C}).$$

L_0 is identified with the submanifold of $L^2((0,\infty);\mathcal{C})$ consisting of all functions having compact support; a net $f_\alpha \in L_0$ converges to $f \in L_0$ iff there is a $T > 0$ such that f_α is supported in $(0,T)$ for sufficiently large α, and

$$\lim_\alpha \int_0^T \|f_\alpha(x) - f(x)\|^2 \, dx = 0.$$

L is isometrically anti-isomorphic to the dual of L_0 by way of the sesquilinear pairing

$$f, g \in L_0 \times L \mapsto \langle f, g\rangle = \int_0^\infty \langle f(x), g(x)\rangle \, dx.$$

A function $t \in (0,\infty) \mapsto \phi_t \in L$ is called *measurable* if it is a Borel function relative to the weak* topology on L, i.e.,

$$t \in (0,\infty) \mapsto \langle f, \phi_t\rangle$$

should be a complex-valued Borel function for every $f \in L_0$.

Definition 3.1. *An additive cocycle is a measurable function $t \in (0,\infty) \mapsto \phi_t \in L$ satisfying*

$$\phi_{s+t}(x) = \phi_s(x) + \phi_t(x+s) \qquad a.e.(dx)$$

for every $s, t > 0$.

The purpose of this section is to prove the following characterization of additive cocycles.

Theorem 3.2. *Let $\{\phi_t : t > 0\}$ be an additive cocycle. Then there is a function $f \in L$ such that*

$$\phi_t(x) = f(x+t) - f(x) \qquad a.e.(dx)$$

for every $t > 0$.

Remarks. Theorem 3.2 bears a resemblance to known results about multiplicative cocycles associated with transitive actions of topological groups on topological spaces [20]. However, our setting here differs in several key aspects. Rather than a group action we have a transitive action of the additive semigroup of positive reals. Moreover, the elements of L must satisfy nontrivial integrability conditions.

Notice too that we make no assumption about the boundedness of $\{\phi_t : t > 0\}$. That is to say, we do *not* assume that

$$\sup_{t>0} |\langle f, \phi_t\rangle| \leq M_f < \infty$$

for every $f \in L_0$. Indeed, it is easy to give examples of cocycles for which this condition is not satisfied. The fact that additive cocycles can be unbounded means that softer techniques that require some form of boundedness (specifically, techniques involving the use of Banach limits on the additive semigroup $\mathbb{R}^+$) are not available. For these reasons we have taken some care to give a complete proof of Theorem 3.2.

Remark on measurability. Every weak* measurable function $t \mapsto \phi_t \in L$ is a Borel mapping of metric spaces. This means that for every $g \in L$, the function

$$t \in (0,\infty) \mapsto d(\phi_t, g)$$

is Borel measurable. To see that, notice that for each $n = 1, 2, \dots,$

$$\|\phi_t - g\|_n = \sup_{k\geq 1} |\, \langle f_k, \phi_t - g\rangle \,|,$$

the supremum extended over a countable set of functions $f_1, f_2, \dots$ which is dense in the unit ball of $L^2((0,n);\mathcal{C})$. Hence $t \mapsto \|\phi_t - g\|_n$ is a Borel function for each n. It follows that

$$d(\phi_t, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|\phi_t - g\|_n}{1 + \|\phi_t - g\|_n}$$

is a Borel function of t.

The following lemma asserts that additive cocycles are metrically continuous. I am indebted to Calvin Moore for pointing out a pertinent reference [12, Théorème 4, p. 23] for a key step in its proof.

Lemma 3.3. *Let $\{\phi_t : t > o\}$ be an additive cocycle. Then $t \mapsto \phi_t$ moves continuously in the metric topology of L and we have*

$$\lim_{t\to 0+} \phi_t = 0.$$

proof. Let $\{T_t : t \geq 0\}$ be the natural translation semigroup which acts in L by

$$T_t f(x) = f(x + t).$$

For every $n = 1, 2, \dots$ and every $f \in L$ we have

$$\lim_{t\to 0} \|T_t f - f\|_n^2 = \lim_{t\to 0} \int_0^n \|f(x+t) - f(x)\|^2 \, dx = 0,$$

and hence $d(T_t f, f) \to 0$ as $t \to 0$. Thus the semigroup $\{T_t : t \geq 0\}$ is continuous in the metric topology of L.

By the preceding remark, the function

$$t \in (0, \infty) \mapsto \phi_t \in L$$

is a Borel function taking values in a separable complete metric space. Thus there is a subset $N \subseteq (0,\infty)$ of the first category in $(0,\infty)$ such that the restriction of this function to $(0,\infty) \setminus N$ is metrically continuous [19, p. 306].

Let $t_0 \in (0,\infty)$ and choose $t_n \in (0,\infty)$ such that $t_n \to t_0$ as $n \to \infty$. Since

$$M = \cup_{n=0}^{\infty} N - t_n$$

is a first category subset of $(0,\infty)$ we may find $s \in (0,\infty) \setminus M$. Then $s + t_n$ belongs to $(0,\infty) \setminus N$ for every $n = 0, 1, 2, \dots,$ and it follows that

$$\lim_{n\to\infty} d(\phi_{s+t_n}, \phi_{s+t_0}) = 0.$$

Writing

$$\phi_{t_0} + T_{t_0}\phi_s = \phi_{t_0+s} = \lim \phi_{t_n+s} = \lim (\phi_{t_n} + T_{t_n}\phi_s)$$

and noting that by the metric continuity of $\{T_t\}$ we have

$$\lim_{n\to\infty} T_{t_n}\phi_s = T_{t_0}\phi_s,$$

we conclude that

$$\phi_{t_0} = \lim_{n\to\infty} \phi_{t_n}.$$

For continuity at $t = 0+$, notice that for every $s, t_0 > 0$ we have

$$\phi_{s+t_0} = \phi_s + T_s\phi_{t_0}.$$

Letting $s \to 0+$ and noting that $\phi_{s+t_0} \to \phi_{s+t_0}$ and $T_s\phi_{t_0} \to \phi_{t_0}$, we obtain

$$\lim_{s\to 0+} \phi_s = 0,$$

as required $\square$

proof of Theorem 3.2. By the Fubini theorem we may find a Borel function

$$v : (t, x) \in (0,\infty)\times(0,\infty) \mapsto v(t, x) \in \mathcal{C}$$

such that $v(t, x) = \phi_t(x)$ almost everywhere with respect to the product measure $dt\,dx$ on $(0,\infty)\times(0,\infty)$. Because of the cocycle condition

$$\phi_{s+t}(x) = \phi_s(x) + \phi_t(x+s) \qquad \text{a.e. } (dx)$$

it follows that v must satisfy

$$v(s+t, x) = v(s, x) + v(t, x+s)$$

for almost every triple $(s, t, x) \in (0,\infty)^3$ with respect to $ds\,dt\,dx$. By the Fubini theorem there is a Borel set $N \subseteq (0,\infty)$ of measure zero such that for every $x \in (0,\infty)\setminus N$ we have

$$v(s+t, x) = v(s, x) + v(t, x+s)$$

almost everywhere $ds\,dt$. Choose a sequence $x_1 > x_2 > \dots$ in $(0,\infty)\setminus N$ which decreases to 0.

We claim that there is a sequence of Borel functions $f_n : (x_n,\infty) \to \mathcal{C}$ satisfying

$$f_n(\xi+t) - f_n(\xi) = v(t, \xi) \tag{3.5.1}$$

almost everywhere on $(x_n,\infty)\times(0,\infty)$ with respect to $d\xi\,dt$, and which is coherent in that

$$f_{n+1}\restriction_{(x_n,\infty)} = f_n \qquad \text{almost everywhere.} \tag{3.5.2}$$

Proceeding inductively, we define f_1 by

$$f_1(\xi) = v(\xi - x_1, x_1), \qquad \xi > x_1$$

To verify (3.5.1), note that for $\lambda, t > 0$ and $\xi = \lambda + x_1$ we have

$$f_1(\lambda + x_1 + t) - f_1(\lambda + x_1) = v(\lambda + t, x_1) - v(\lambda, x_1)$$

which by (3.4) is almost everywhere ($d\lambda\, dt$) equal to $v(t, x_1 + \lambda)$, hence we have (3.5.1).

Assuming now that f_k has been defined for $1 \leq k \leq n$, define $g : (x_{n+1}, \infty) \to \mathcal{C}$ by

$$g(\xi) = v(\xi - x_{n+1}, x_{n+1}).$$

If we replace x_1 with x_{n+1} in the argument of the preceding paragraph we obtain the conclusion

$$g(\xi + t) - g(\xi) = v(t, \xi)$$

almost everywhere ($d\xi\, dt$) on $(x_{n+1}, \infty) \times (0, \infty)$. Restricting ξ to the interval $(x_n, \infty) \subseteq (x_{n+1}, \infty)$ then gives

$$g(\xi + t) - g(\xi) = v(t, \xi) = f_n(\xi + t) - f_n(\xi)$$

almost everywhere ($d\xi\, dt$) on $(x_n, \infty) \times (0, \infty)$. Thus the function $h : (x_n, \infty) \to \mathcal{C}$ defined by $h(x) = g(x) - f_n(x)$ is Borel-measurable and translation invariant in the sense that for almost every $t \geq 0$ we have

$$h(\xi + t) = h(\xi) \qquad \text{a.e. } (d\xi).$$

It follows that there is a vector $c \in \mathcal{C}$ such that

$$h(\xi) = c \qquad \text{a.e. } (d\xi).$$

If we set $f_{n+1} = g(x) - c$ then we obtain both required conditions (3.5.1) and (3.5.2).

Because of the coherence property (3.5.2), there is a Borel function $f : (0, \infty) \to \mathcal{C}$ satisfying

$$f(\xi) = f_n(\xi) \qquad \text{a.e. } (d\xi)$$

on (x_n, ∞), for every $n = 1, 2, \dots$; and because of (3.5.1) we have

$$f(\xi + t) - f(\xi) = v(t, \xi)$$

for almost every pair $(\xi, t) \in (0, \infty) \times (0, \infty))$ with respect to the product measure $d\xi\, dx$.

It follows that

$$f(\xi + t) - f(\xi) = \phi_t(\xi)$$

almost everywhere $d\xi\, dt$, and by another application of Fubini's theorem we may conclude that there is a Borel set $N \subseteq (0, \infty)$ such that for all $t \in (0, \infty) \setminus N$ we have

$$\phi_t(x) = f(x + t) - f(x) \qquad \text{a.e. } (dx). \tag{3.6}$$

We show next that the exceptional set N of (3.6) can be eliminated. To that end, consider the vector space $\mathcal{F}$ consisting of all Borel functions

$$F : (0, \infty) \to \mathcal{C}$$

where we make the traditional identification of two functions that agree almost everywhere. We endow this space with the topology of local convergence in measure. More precisely, a net $f_\alpha \in \mathcal{F}$ converges to $F \in \mathcal{F}$ iff for every $T > 0$, the restrictions $F_\alpha \restriction_{(0,T)}$ converge in Lebesgue measure to $F \restriction_{(0,T)}$. $\mathcal{F}$ is metrizable as a separable complete metric space by way of

$$d(F,G) = \sum_{n=1}^{\infty} 2^{-n} d_n(F,G)$$

where for $n = 1, 2, \dots,$

$$d_n(F,G) = \int_0^n \frac{\|F(x) - G(x)\|}{1 + \|F(x) - G(x)\|}\, dx.$$

The translation semigroup $\{T_t : t \geq 0\}$ defined by

$$T_t F(x) = F(x+t)$$

acts continuously on the Frèchet space $\mathcal{F}$.

It follows from these remarks that the right side of (3.6) is continuous in t, provided that we consider $T_t f - f$ as an element of $\mathcal{F}$. Since the inclusion map $L \subseteq \mathcal{F}$ is continuous, it follows from Lemma 3.3 that the left side of (3.6) defines a continuous function

$$t \in (0,\infty) \mapsto \phi_t \in \mathcal{F}.$$

Equation (3.6) implies that these two continuous functions agree on the complement of a null set, and hence they agree for all $t > 0$.

Thus, we have a Borel function $f : (0,\infty) \to L$ with the property that for every $t > 0$,

$$\phi_t(x) = f(x+t) - f(x) \qquad \text{a.e. } (dx). \tag{3.7}$$

It remains to show that the condition $\phi_t \in L$ for every $t > 0$ implies that f itself belongs to L. We will deduce that from the following result, the proof of which is based on an argument shown to me by Henry Helson, who kindly consented to its inclusion in this paper.

Lemma 3.8. *Let f be a nonnegative Borel function defined on $(0,\infty)$ satisfying*

$$\int_0^T |f(x+t) - f(x)|^2\, dx < \infty$$

for every $T > 0$ and every $t > 0$. Then for every $T > 0$ we have

$$\int_0^T |f(x)|^2\, dx < \infty.$$

proof. Fix $T > 0$. It suffices to show that for every nonnegative funtion $g \in L^2(0,T)$ we have

$$\int_0^T f(x) g(x)\, dx < \infty.$$

To that end, find a function $u \in L^2(0,1)$ such that $u(x) > 0$ for every x and

$$\int_0^1 f(x)u(x)\,dx < \infty,$$

and define a function $\Phi : (0,\infty) \to \mathbb{R}^+$ by

$$\Phi(t) = \int_0^1 |f(x+t) - f(x)|u(x)\,dx.$$

We claim that Φ is continuous and tends to 0 as $t \to 0+$. Indeed, we may apply Lemma 3.3 to the additive cocycle

$$\phi_t(x) = f(x+t) - f(x)$$

in the case where $\mathcal{C}$ is the complex numbers to conclude that $t \in (0,\infty) \mapsto \phi_t \in L$ is metrically continuous and tends to 0 as $t \to 0+$. Since $F \in L \mapsto |F| \in L$ is clearly continuous, the same is true of the modulus $t \in (0,\infty) \mapsto |\phi_t| \in L$, and the claim follows.

Notice next that for every positive $g \in L^2$,

$$\int_0^1 \int_0^T f(x+t)g(t)u(x)\,dt\,dx < \infty. \tag{3.10}$$

Indeed, since

$$f(x+t)g(t)u(x) \le |\phi_t(x)|g(t)u(x) + f(x)g(t)u(x),$$

and since

$$\int_0^1 \int_0^T |\phi_t(x)|g(t)u(x)\,dt\,dx = \int_0^T \Phi(t)g(t)\,dt < \infty$$

and

$$\int_0^1 \int_0^T f(x)g(t)u(x)\,dt\,dx = \int_0^T f(x)u(x)\,dx \int_0^1 g(t)\,dt < \infty,$$

(3.10) follows.

Since $u > 0$, (3.10) implies that

$$\int_0^T f(x+t)g(t)\,dt < \infty$$

for almost every $x \in (0,1)$. Nor for every $x > 0$ we have

$$\int_0^T f(t)g(t)\,dt \le \int_0^T f(x+t)g(t)\,dt + \int_0^T |f(x+t) - f(x)|g(t)\,dt.$$

The first integral on the right is finite for certain values of $x \in (0,1)$ and the second one is finite for all $x > 0$. Thus (3.9) follows. $\square$

To complete the proof of theorem 3.2, we must show that any measurable function $f : (0,\infty) \to \mathcal{C}$ for which the differences $\phi_t(x) = f(x+t) - f(x)$ belong to

L must itself belong to L. Equivalently, if $f : (0, \infty) \to \mathcal{C}$ is a Borel function for which

$$\int_0^T \|f(x+t) - f(x)\|^2 \, dx < \infty$$

for every $T > 0$ and every $t > 0$, then

$$\int_0^T \|f(x)\|^2 \, dx < \infty \tag{3.11}$$

for every $T > 0$.

To see that, fix such an f and consider $F(x) = \|f(x)\|$. We have

$$|F(x+t) - F(x)| \leq \|f(x+t) - f(x)\|,$$

hence F satisfies the hypotheses of Lemma 3.8. It follows from Lemma 3.8 that F is locally in L^2. $\square$

4. Classification of additive forms. Before stating the main result on classification of metric path spaces we introduce the concept of a strongly spanning set. Let H be a Hilbert space and let S be a subset of H. We will write e^H for the symmetric Fock space over the one-particle space H,

$$e^H = \sum_{n=0}^{\infty}{}^{\oplus} H^{(n)},$$

$H^{(n)}$ denoting the symmetric tensor product of n copies of H when $n \geq 1$, and where $H^{(0)}$ is defined as $\mathbb{C}$. Consider the exponential map $\exp : H \to e^H$, defined by

$$\exp(\xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \, \xi^n,$$

ξ^n denoting $\xi^{\otimes n}$ if $n \geq 1$ and $\xi^0 = 1 \in \mathbb{C}$. It is well known that e^H is spanned by

$$\exp(H) = \{\exp(\xi) : \xi \in H\}.$$

Definition 4.1. *A subset $S \subseteq H$ is said to strongly span H if e^H is spanned by the set of vectors* $\exp(S) = \{exp(\xi) : \xi \in S\}$.

Remarks. When we use the term 'span', we of course mean *closed* linear span. Every strongly spanning set must span H, but the converse if false, as the following remarks show. In general, every vector $\zeta \in e^H$ give rise to a holomorphic function $f_\zeta : H \to \mathbb{C}$ by way of

$$f_\zeta(\xi) = \langle \exp(\xi), \zeta \rangle \, .$$

Indeed, if we let ζ_n be the projection of ζ onto $H^{(n)}$, then we have a representation of f_ζ as a power series

$$f_\zeta(\xi) = \sum_0^{\infty} \frac{1}{\sqrt{n!}} \, \langle \xi^n, \zeta_n \rangle \, ,$$

and because $\sum_n \|\zeta_n\|^2 = \|\zeta\|^2 < \infty$, this power series converges absolutely and uniformly over the ball in H of radius R for every $R > 0$. Thus, $\mathcal{T} = \{f_\zeta : \zeta \in e^H\}$

is a complex vector space of entire functions defined on H. S is a strongly spanning set iff

$$F \in \mathcal{F}, \quad F(S) = \{0\} \implies F = 0.$$

If H is finite dimensional then every holomorphic polynomial $F : H \to \mathbb{C}$ belongs to $\mathcal{F}$. In particular, if H is two dimensional and $\{e_1, e_2\}$ is an orthonormal basis for H then

$$S = \{\lambda(e_1 + e_2) : \lambda \in \mathbb{C}\} \cup \{\lambda(e_1 - e_2) : \lambda \in \mathbb{C}\}$$

clearly spans H because it contains $e_1 + e_2$ and $e_1 - e_2$. On the other hand, S is not strongly spanning because

$$F(\xi) = \langle \xi, e_1 \rangle^2 - \langle \xi, e_2 \rangle^2$$

is a nonzero polynomial which vanishes on S. More generally, any spanning subset S of a finite dimensional H which is contained in the zero set of a nontrivial polynomial will fail to be strongly spanning. This will be the case whenever S is an algebraic set, or an algebraic variety in H.

The following is our main classification of metric path spaces.

Theorem 4.3. *Let P be a path space and let $g : P^2 \to \mathbb{C}$ be a separable measurable additive form. Then there is a separable Hilbert space $\mathcal{C}$, a complex-valued function $\rho : P \to \mathbb{C}$ and a mapping of fiber spaces*

$$\log : P \to \mathcal{P}_{\mathcal{C}}$$

such that $\log(xy) = \log(x) \boxplus \log(y)$ *for every $x, y \in P$, and such that for every $t > 0$, $x_1, x_2 \in P(t)$,*

(4.3.1) $\log(P(t))$ *strongly spans* $\mathcal{P}_{\mathcal{C}}(t)$

(4.3.2) $g(x_1, x_2) = \langle \log(x_1), \log(x_2) \rangle + \rho(x_1) + \overline{\rho(x_2)}$

Remarks. The assertion that log is a mapping of fiber spaces means $\log(P(t)) \subseteq \mathcal{P}_{\mathcal{C}}(t)$ for every $t > 0$. Thus, log defines a homomorphism of the path space structure of P into that of $\mathcal{P}_{\mathcal{C}}$. Property 4.3.1 asserts that, even though $\log(P(t))$ may not be dense in $\mathcal{P}_{\mathcal{C}}(t)$, it is a rich enough subset so that

$$\overline{\operatorname{span}} \exp(P(t)) = e^{\mathcal{P}_{\mathcal{C}}(t)}.$$

Finally, notice that property 4.3.2 implies that g is an *exact* form in the sense of (2.5). Indeed, if we take $x_1, x_2 \in P(s)$ and $y_1, y_2 \in P(t)$ then by definition of the concatenation operation $\boxplus$ in $\mathcal{P}_{\mathcal{C}}$ we have

$$\langle \log(x_1) \boxplus \log(y_1), \log(x_2) \boxplus \log(y_2) \rangle = \langle \log(x_1), \log(x_2) \rangle + \langle \log(y_1), \log(y_2) \rangle .$$

Using the fact that $\log(x_k y_k) = \log(x_k) \boxplus \log(y_k)$ for $k = 1, 2$ we find that

$$g(x_1 y_1, x_2 y_2) - g(x_1, x_2) - g(y_1, y_2) = \psi(x_1, y_1) + \overline{\psi(x_2, y_2)}$$

where $\psi(x, y) = \rho(xy) - \rho(x) - \rho(y)$ and ρ is the function given by 4.3.2

The proof of Theorem 4.3 will occupy most of the remainder of this section, and will proceed along the following lines. We first use g to associate a Hilbert space H_t with $P(t)$ for every $t > 0$. We then show that for $s < t$, H_s embeds naturally in H_t so that we can form an inductive limit of Hilbert spaces

$$H_\infty = \lim_{\rightarrow} H_t.$$

We introduce a strongly continuous semigroup of isometries acting in H_∞ which will turn out to be *pure*. This implies that H_∞ can be coordinatized in such a way that it becomes an L^2 space of vector valued functions

$$H_\infty = L^2((0,\infty);\mathcal{C})$$

and the semigroup of isometries is the natural shift semigroup. Finally, we use the results of section 3 to solve a cohomological problem. Once that is accomplished we can define the required "logarithm" $\log : P \to \mathcal{P}_\mathcal{C}$ and verify its properties.

Definition of H_t. Fix $t > 0$. Let $\mathbb{C}_0 P(t)$ denote the complex vector space of all finitely nonzero functions $f : P(t) \to \mathbb{C}$ satisfying the condition $\sum_x f(x) = 0$, and let $\langle \cdot, \cdot \rangle$ be the sesquilinear form defined on $\mathbb{C}_0 P(t)$ by

$$\langle f, g\rangle = \sum_{x,y\in P(t)} f(x)\overline{g(y)}g(x,y).$$

$\langle \cdot, \cdot \rangle$ is positive semidefinite, and after passing to the quotient of $\mathbb{C}_0 P(t)$ by the subspace of null functions $K_t = \{f : \langle f, f\rangle = 0\}$ we obtain an inner product space, whose completion is denoted H_t.

Now $\mathbb{C}_0 P(t)$ is spanned by the set of all differences $\{\delta_x - \delta_y : x, y \in P(t)\}$, δ_z denoting the unit function

$$\delta_z(u) = \begin{cases} 1, & \text{if } u = z \\ 0, & \text{otherwise.} \end{cases}$$

Hence H_t is spanned by the set $\{[x] - [y] : x, y \in P(t)\}$, where $[x] - [y]$ denotes the element $\delta_x - \delta_y + K_t$. The inner product in H_t is characterized by

$$\langle [x_1] - [y_1], [x_2] - [y_2]\rangle = g(x_1, x_2) - g(x_1, y_2) - g(y_1, x_2) + g(y_1, y_2), \tag{4.4}$$

for $x_1, x_2, y_1, y_2 \in P(t)$. Notice that although we have written $[x] - [y]$ as if it were a difference, it is not actually the difference it appears to be since $[x]$ and $[y]$ do not belong to H_t. It is in fact a two-variable function which satisfies a certain cocycle identity. But the notation is convenient provided one is careful never to treat $[x]$ and $[y]$ as if they were elements of H_t.

H_t is separable because of our separability hypothesis on g (See the discussion preceding Definition 2.2). When it is necessary to distinguish between the various inner products we will write $\langle \cdot, \cdot \rangle_t$ for the inner product on H_t.

Embedding H_s in H_t for $s < t$. Fix s, t with $0 < s < t$ and choose an element $e \in P(t-s)$. We want to show that there is an isometric linear map of H_s into H_t which carries differences of the form $[x_1]-[x_2]$ with $x_i \in P(s)$ into $[x_1e]-[x_2e]$, and moreover that this isometry does not depend on the particular choice of $e \in P(t-s)$. To that end, we claim that for all $x_i, y_i \in P(s)$, and $z_i \in P(t-s)$ $i = 1, 2$, we have

$$\langle [x_1e] - [x_2e], [y_1z_1] - [y_2z_2] \rangle_t = \langle [x_1] - [x_2], [y_1] - [y_2] \rangle_s \tag{4.5}$$

Indeed, because of (4.4) the left side is

$$g(x_1e, y_1z_1) - g(x_1e, y_2z_2) - g(x_2e, y_1z_1) + g(x_2e, y_2z_2) \tag{4.6}$$

But by (2.4) we have,

$$g(x_ie, y_jz_j) = g(x_i, y_j) + g(e, z_j) + \psi(x_i, e) + \overline{\psi(y_j, z_j)}.$$

It follows that for $j = 1, 2$,

$$g(x_1e, y_jz_j) - g(x_2e, y_jz_j) = g(x_1, y_j) - g(x_2, y_j) + \psi(x_1, e) - \psi(x_2, e).$$

When we subtract this expression for $j = 2$ from the expression for $j = 1$ the terms involving ψ cancel and we are left with

$$g(x_1, y_1) - g(x_1, y_2) - g(x_2, y_1) + g(x_2, y_2)$$

which is the right side of (4.5).

By taking $z_1 = z_2 = e$ in (4.5) we see that

$$\langle [x_1e] - [x_2e], [y_1e] - [y_2e] \rangle_t = \langle [x_1] - [x_2], [y_1] - [y_2] \rangle_s$$

and hence there is a unique linear isometry $V(t, s) : H_s \to H_t$ such that

$$V(t, s)([x_1] - [x_2]) = [x_1e] - [x_2e]$$

for every $x_1, x_2 \in P(s)$. Moreover, since H_t is spanned by elements of the form $[y_1z_1] - [y_2z_2]$ for $y_i \in P(s)$, $z_i \in P(t-s)$ (here we use the fact that $P(t) = P(s)P(t-s)$), it also follows from (4.5) that $V(t, s)$ is independent of the particular choice of $e \in P(t-s)$; in more concrete terms, if e and f are two elements of $P(t-s)$, then for all $x_1, x_2 \in P(s)$ we have

$$[x_1e] - [x_2e] = [x_1f] - [x_2f].$$

It follows from the latter that we have the consistency relation

$$V(t, s)V(s, r) = V(t, r)$$

for all $0 < r < s < t$. Indeed, if we choose $e_1 \in P(r-s)$ and $e_2 \in P(t-s)$ then for $x_1, x_2 \in P(r)$ we have

$$V(t, s)V(s, r)([x_1] - [x_2]) = V(t, s)([x_1e_1] - [x_2e_1]) = [x_1e_1e_2] - [x_2e_1e_2]$$

and the right side is $V(t,r)([x_1]-[x_2])$ simply because e_1e_2 is an element of $P(t-s)$.

Thus we can form the inductive limit of inner product spaces

$$\lim_{\rightarrow} H_t.$$

Explicitly, this consists of all functions $t \in (0,\infty) \mapsto \xi_t \in H_t$ having the property that there is a $T = T_\xi > 0$ such that for all $t > T$,

$$\xi_t = V(t,T)\xi_T.$$

The inner product in the inductive limit is defined by

$$\langle \xi, \eta \rangle = \lim_{t\to\infty} \langle \xi_t, \eta_t \rangle_t .$$

H_∞ is defined as the completion of

$$\lim_{\rightarrow} H_t,$$

and it is a separable Hilbert space.

Choose $x_1, x_2 \in P(t)$. By a slight abuse of notation we will write $[x_1] - [x_2]$ for the element of H_∞ defined by the function

$$\xi_\lambda = \begin{cases} 0, & \text{for } \lambda \le t \\ V(\lambda,t)([x_1]-[x_2]), & \text{for } \lambda > t. \end{cases}$$

Notice that *by definition* of H_∞, we will have

$$[x_1e] - [x_2e] = [x_1] - [x_2]$$

for every $x_1, x_2 \in P(t)$, $t > 0$ and for an arbitrary element e of P. H_∞ is spanned by the set of formal differences

$$\{[x_1] - [x_2] : x_i \in P(t), t > 0\}.$$

Finally, note that the inner product in H_∞ is defined by its values on these formal differences as follows. Choose $s \neq t$, $x_1, x_2 \in P(s)$ and $y_1, y_2 \in P(t)$. In order to evaluate the inner product $\langle [x_1] - [x_2], [y_1] - [y_2] \rangle$ we may suppose that $s < t$. Choose any $e \in P(t-s)$. Then since $[x_1] - [x_2] = [x_1e] - [x_2e]$ we have

$$\begin{aligned} \langle [x_1] - [x_2], [y_1] - [y_2] \rangle &= \langle [x_1e] - [x_2e], [y_1] - [y_2] \rangle \\ &= g(x_1e, y_1) - g(x_1e, y_2) - g(x_2e, y_1) + g(x_2e, y_2). \end{aligned}$$

The subspaces $N_t \subseteq H_\infty$. For every $t > 0$ we define a subspace N_t of H_∞ as follows

$$N_t = \overline{\text{span}}\{[x_1] - [x_2] : x_i \in P(t)\}.$$

Choose $0 < s < t$ and $x_1, x_2 \in P(s)$. The preceding remarks imply that the element of H_∞ represented by the difference $[x_1] - [x_2]$ can be identified with a difference $[y_1] - [y_2]$ of elements from $P(t)$ by taking $y_i = x_ie$ for some $e \in P(t-s)$. It follows that the spaces N_t are increasing

(4.7.1.) $$0 \le s \le t \implies N_s \subseteq N_t$$

Moreover, since the images of the spaces H_t, $t > 0$ in H_∞ span H_∞ we also have

(4.7.2) $$\overline{\cup_{t>0} N_t} = H_\infty.$$

The semigroup $\{U_t : t \geq 0\}$. We now introduce a semigroup of isometries $\{U_t : t \geq 0\}$ acting on H_∞. Fix $t > 0$ and choose an element $e \in P(t)$.

There is a formula analogous to (4.5) in which the order of multiplication is reversed. That is, for $0 < s < t$, $x_i, y_i \in P(s)$ and $e, z_i \in P(t-s)$ we claim that

$$\langle [ex_1] - [ex_2], [z_1 y_1] - [z_2 y_2] \rangle_t = \langle [x_1] - [x_2], [y_1] - [y_2] \rangle_s \tag{4.8}$$

Notice that the inner product on the left (resp. right) is taken in the Hilbert space H_t (resp. H_s). The proof of (4.8) is the same as the proof of (4.5). The identity (4.8) implies that if we choose an element $f \in P(t)$ for some $t > 0$ then for any $s > 0$ and any pair of elements $x_1, x_2 \in P(s)$, the element of H_∞ defined by $[fx_1] - [fx_2]$ does not depend on the particular choice of $f \in P(t)$ in that for every $g \in P(t)$ we have

$$[fx_1] - [fx_2] = [gx_1] - [gx_2]. \tag{4.9}$$

Note that in (4.9) the vectors on both sides belong to H_∞. Moreover, if y_1, y_2 is a second pair of elements in $P(s)$ then (4.8) also implies

$$\langle [fx_1] - [fx_2], [fy_1] - [fy_2] \rangle = \langle [x_1] - [x_2], [y_1] - [y_2] \rangle .$$

Now fix $t > 0$ and choose $f \in P(t)$. It follows that there is a unique isometry $U_{t,s} : N_s \to N_{s+t}$ which satisfies

$$U_{t,s}([x_1] - [x_2]) = [fx_1] - [fx_2].$$

Because of (4.9), $U_{t,s}$ does not depend on the choice of f. Notice too that $U_{t,s}$ does not depend on s. Indeed, if $0 < s_1 < s_2$ and $x_1, x_2 \in P(s_1)$ then for any element $z \in P(s_2 - s_1)$ we have

$$U_{t,s_1}([x_1] - [x_2]) = [fx_1] - [fx_2] = [(fx_1)z] - [(fx_2)z],$$

while

$$U_{t,s_2}([x_1] - [x_2]) = U_{t,s_2}([x_1 z] - [x_2 z]) = [f(x_1 z)] - [f(x_2 z)].$$

The right hand sides of these two formulas agree because of the associativity of the multiplication in P.

By (4.7.2) there is a unique isometry $U_t : H_\infty \to H_\infty$ satisfying

$$U_t([x_1] - [x_2]) = [fx_1] - [fx_2]$$

for all $x_1, x_2 \in P(s)$, $s > 0$. For $t = 0$ we set $U_0 = \mathbf{1}$. Finally, note that $\{U_t : t \geq 0\}$ is a semigroup. Indeed, given $s, t > 0$ we choose $f \in P(s)$ and $g \in P(t)$, and note that for every $x_1, x_2 \in P(r)$ we have

$$U_s U_t([x_1] - [x_2]) = U_s([gx_1] - [gx_2]) = [fgx_1] - [fgx_2].$$

The right side must be $U_{s+t}([x_1] - [x_2])$ because the product fg belongs to $P(s+t)$. This shows that $U_s U_t = U_{s+t}$ on each N_r, and by (4.7.2) it follows that $U_s U_t = U_{s+t}$.

Strong continuity. Since H_∞ is a separable Hilbert space, the strong continuity of $\{U_t : t \geq 0\}$ will follow is we prove that the operator function

$$t \in (0, \infty) \mapsto U_t \in \mathcal{B}(H_\infty)$$

is weakly measurable in the sense that $\langle U_\lambda \xi, \eta \rangle$ defines a Borel function on $0 < \lambda < \infty$ for every $\xi, \eta \in H_\infty$ [2, Proposition 2.5 (ii)]. In turn, because of the semigroup property it suffices to verify this for λ restricted to the interval $0 < \lambda \leq 1$. To that end, we claim that for any pair of vectors ξ_1, ξ_2 in the spanning set

$$\bigcup_{t>0} \{[x_1] - [x_2] : x_i \in P(t)\},$$

the function

$$\lambda \in (0, 1] \mapsto \langle U_\lambda \xi_1, \xi_2 \rangle \tag{4.10}$$

is Borel measurable.

To see this suppose that $\xi = [x_1] - [x_2]$ with $x_i \in P(s)$ and $\xi_2 = [y_1] - [y_2]$ with $y_i \in P(t)$. By replacing y_1, y_2 with $y_1 v, y_2 v$ for an appropriate $v \in P$ we may assume that t is as large as we please, and in particular we may assume that $t > s + 1$. Choosing elements $f \in P(t - s - \lambda)$ and $e \in P(\lambda)$, we have

$$U_\lambda([x_1] - [x_2]) = U_\lambda([x_1 f] - [x_2 f]) = [e x_1 f] - [e x_2 f],$$

and thus

$$\langle U_\lambda \xi_1, \xi_2 \rangle = \langle [e x_1 f] - [e x_2 f], [y_1] - [y_2] \rangle = \alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22}, \tag{4.11}$$

where

$$\alpha_{ij} = g(e x_i f, y_j).$$

In order to calculate the terms α_{ij} we make use of the propagators of y_2 and y_2 to obtain the factorizations

$$y_j = y_j(0, \lambda) y_j(\lambda, \lambda + t) = y_j(0, \lambda) y_j(\lambda, \lambda + s) y_j(\lambda + s, t).$$

Thus

$$\begin{aligned}
\alpha_{ij} =& g(e x_i f, y_j(0, \lambda) y_j(\lambda, \lambda + t)) \\
=& g(e, y_j(0, \lambda)) + g(x_i f, y_j(\lambda, \lambda + t)) + \psi(e, x_i f) + \overline{\psi(y_j(0, \lambda), y_j(\lambda, \lambda + t))} \\
=& g(e, y_j(0, \lambda)) + g(x_i, y_j(\lambda, \lambda + s)) + g(f, y_j(\lambda + s, t)) + \\
& \psi(x_i, f) + \overline{\psi(y_j(\lambda, \lambda + s), y_j(\lambda + s, t))} + \psi(e, x_i f) + \overline{\psi(y_j(0, \lambda), y_j(\lambda, \lambda + t))}.
\end{aligned}$$

Noting that α_{ij} has the form

$$\alpha_{ij} = g(x_i, y_j(\lambda, \lambda + s)) + u_i + v_j$$

for appropriate complex numbers u_1, u_2, v_1, v_2 (which depend on λ), it follows that the u's and v's cancel out of the right side of (4.11) and we are left with

$$\begin{aligned}
\alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22} =& g(x_1, y_1(\lambda, \lambda + s)) - g(x_2, y_1(\lambda, \lambda + s)) + \\
& g(x_2, y_2(\lambda, \lambda + s)) - g(x_1, y_2(\lambda, \lambda + s)).
\end{aligned}$$

Since g is a measurable form, each of the two functions

$$\lambda \in (0, 1] \mapsto g(x_1, y_j(\lambda, \lambda + s)) - g(x_2, y_j(\lambda, \lambda + s))$$

$j = 1, 2$ is a Borel function, and thus the right side of the previous formula is a difference of Borel functions.

Purity of $\{U_t : t \geq 0\}$. We claim next that the semigroup $\{U_t : t \geq 0\}$ is *pure* in the sense that

$$\bigcap_{t>0} U_t H_\infty = \{0\}. \tag{4.12}$$

This is a consequence of (4.7.1), (4.7.2) and the following.

Lemma 4.13. *For every $t > 0$, H_∞ decomposes into a direct sum*

$$H_\infty = N_t \oplus U_t H_\infty.$$

proof. Fix $t > 0$. We show first that N_t is orthogonal to $U_t H_\infty$. For that, it suffices to show that for every $r > 0$ and for $x_1, x_2 \in P(t)$, $y_1, y_2 \in P(r)$ we have

$$\langle [x_1] - [x_2], U_t([y_1] - [y_2])\rangle = 0. \tag{4.14}$$

Choose elements $e \in P(t)$, $f \in P(r)$. Then we have

$$\begin{aligned} [x_1] - [x_2] &= [x_1 f] - [x_2 f], \qquad \text{and} \\ U_t([y_1] - [y_2]) &= [ey_1] - [ey_2]. \end{aligned}$$

Thus the left side of (4.14) has the form

$$\langle [x_1 f] - [x_2 f], [ey_1] - [ey_2]\rangle = \alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22},$$

where

$$\alpha_{ij} = g(x_i f, ey_j).$$

Using the definition of additive forms (2.3) we have

$$\alpha_{ij} = g(x_i, e) + g(f, y_j) + \psi(x_i, f) + \overline{\psi(e, y_j)} = u_i + v_j$$

where

$$\begin{aligned} u_i &= g(x_i, e) + \psi(x_i, f) \\ v_j &= g(f, y_j) + \overline{\psi(e, y_j)}. \end{aligned}$$

It follows that all of the u's and v's cancel and we are left with the required formula

$$\alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22} = 0.$$

To show that $N_t \cup U_t H_\infty$ spans H_∞ it is enough to show that for every $r > t$ and every pair $x_1, x_2 \in P(r)$, we have

$$[x_1] - [x_2] \in N_t + U_t H_\infty.$$

If we factor $x_i = a_i b_i$ where $a_i = x_i(0, t) \in P(t)$ and $b_i = x_i(t, r) \in P(r - t)$, then we have

$$\begin{aligned} [x_1] - [x_2] = [a_1 b_1] - [a_2 b_2] &= ([a_1 b_1] - [a_2 b_1]) + ([a_2 b_1] - [a_2 b_2]) \\ &= ([a_1] - [a_2]) + U_t([b_1] - [b_2]) \end{aligned}$$

and the right side clearly belongs to $N_t + U_t H_\infty$ $\square$

$\{U_t : t \geq 0\}$ **as a shift.** A familiar theorem asserts that every strongly continuous pure semigroup of isometries is unitarily equivalent to a direct sum of copies of the semigroup of simple unilateral shifts acting on $L^2(0, \infty)$. From this it follows that we can replace H_∞ with the Hilbert space $L^2((0, \infty); \mathcal{C})$ of all square integrable vector valued measurable functions $\xi : (0, \infty) \to \mathcal{C}$ with inner product

$$\langle \xi, \eta \rangle = \int_0^\infty \langle \xi(x), \eta(x) \rangle \, dx$$

in such a way that $\{U_t : t \geq 0\}$ becomes the semigroup

$$U_t \xi(x) = \begin{cases} \xi(x - t), & x > t \\ 0, & 0 < x \leq t. \end{cases}$$

After making this identification, we find that the range of U_t consists of all functions $\xi \in L^2((0, \infty); \mathcal{C})$ which vanish almost everywhere in the interval $0 < x \leq t$. From Lemma 4.13 we conclude that for every $t > 0$,

$$\overline{\text{span}}\{[x_1] - [x_2] : x \in P(t)\} = L^2((0, t); \mathcal{C}), \tag{4.14}$$

the right side denoting the subspace of all functions $\xi \in L^2((0, \infty); \mathcal{C})$ which vanish almost everywhere outside the interval $0 < x \leq t$. Since H_∞ is a separable Hilbert space, it follows that $\mathcal{C}$ must be separable as well.

Finally, we remind the reader that the rules for left and right multiplication in these new "coordinates" are the same as they were in H_∞: if $x_1, x_2 \in P(s)$ and u is any element of $P(t)$, then

$$[x_1 u] - [x_2 u] = [x_1] - [x_2], \qquad \text{and}$$
$$[u x_1] - [u x_2] = U_t([x_1] - [x_2]).$$

Equation (4.14) identifies the space $N_t = \overline{\text{span}}\{[x_1] - [x_2] : x_i \in P(t)\}$ with the space $\mathcal{P}_\mathcal{C}(t)$ of example 2.9 for every $t > 0$. It remains to define the "logarithm" mapping $\log : P \to \mathcal{P}_\mathcal{C}$ with the properties asserted in Theorem 4.3.

Definition of the logarithm. In order to define the logarithm we must first show that certain 2-cocycles are trivial. These cocycles are associated with globally defined left-coherent sections

$$t \in (0, \infty) \mapsto e_t \in P(t)$$

whose existence is established in the following.

Lemma 4.15. *Choose any element $e \in P(1)$. Then there is a left-coherent family of elements $\{e_t : t > 0\}$ with the property that $e_1 = e$.*

proof. For $0 < t \leq 1$ we set $e_t = e(0, t)$ where $\{e(s, t) : 0 \leq s < t \leq 1\}$ is the propagator associated with e as in section 1. For $n < t \leq n + 1$ we set

$$e_t = e_1^n e(0, t - n).$$

It is clear that $\{e_t : t > 0\}$ has the required properties □

Choose such a section $\{e_t : t > 0\}$, which will be fixed throughout the remainder of this section. Define a function $\Gamma : (0, \infty) \times (0, \infty) \to L^2((0, \infty); \mathcal{C})$ by

$$\Gamma(s, t) = [e_s e_t] - [e_{s+t}].$$

We will see presently that Γ is an additive 2-cocycle in the sense that for all $r, s, t > 0$ we have

$$\Gamma(r + s, t) - \Gamma(r, s + t) - U_r \Gamma(s, t) + \Gamma(r, s) = 0,$$

see (4.20). The following asserts that Γ is exact in the sense that we require

Theorem 4.16. *There is a measurable function $t \in (0,\infty) \mapsto \phi_t \in L^2((0,\infty);\mathcal{C})$ such that*

$$\phi_t(x) = 0 \qquad \text{a.e. outside } 0 < x \leq t, \tag{4.16.1}$$

$$\Gamma(s,t) = \phi_{s+t} - \phi_s - U_s\phi_t, \qquad s,t > 0. \tag{4.16.2}$$

Remark 4.17. The measurability assertion of Theorem 4.16 simply means that for every $\xi \in L^2((0,\infty);\mathcal{C})$,

$$t \in (0,\infty) \mapsto \langle \phi_t, \xi \rangle$$

is a complex-valued Borel function. Because $L^2((0,\infty);\mathcal{C})$ is separable, this is equivalent to measurability of $t \mapsto \phi_t$ relative to the metric topology of $L^2((0,\infty);\mathcal{C})$. Notice too that (4.16.1) asserts that $\phi_t \in \mathcal{P}_{\mathcal{C}}(t)$ for every $t > 0$.

Assume, for the moment, that Theorem 4.16 has been proved. We can then define a fiber map

$$\log : P \to \mathcal{P}_{\mathcal{C}}$$

as follows. For $z \in P(t)$, $t > 0$ we put

$$\log(z) = [z] - [e_t] - \phi_t.$$

Notice that $\log(z) \in \mathcal{P}_{\mathcal{C}}(t)$ because both $[z] - [e_t]$ and ϕ_t belong to $\mathcal{P}_{\mathcal{C}}(t)$. For $x, y \in P$ we claim:

$$\log(xy) = \log(x) \boxplus \log(y). \tag{4.18}$$

This is to say that, if $x \in P(s)$ and $y \in P(t)$ then

$$\log(xy) = \log(x) + U_s(\log(y));$$

equivalently,

$$[xy] - [e_{s+t}] - \phi_{s+t} = [x] - [e_s] - \phi_s + U_s([y] - [e_t] - \phi_t).$$

To see that this is the case, note that

$$\begin{aligned}[xy] - [e_{s+t}] =& ([xy] - [e_s y]) + ([e_s y] - [e_s e_t]) + ([e_s e_t] - [e_{s+t}]) \\ =& [x] - [e_s] + U_s([y] - [e_t]) + \Gamma(s,t).\end{aligned}$$

Using (4.16.2) to substitute for $\Gamma(s,t)$ and subtracting ϕ_{s+t} from both sides, we obtain (4.18).

proof of Theorem 4.16. The argument will proceed as follows. We first find a family $\{u_t : t > 0\}$ of Borel functions

$$u_t : (0,\infty) \to \mathcal{C}$$

which satisfy

$$\Gamma(s,t) = u_{s+t} - u_s - U_s(u_t)$$

almost everywhere, for every $s,t>0$. The family $\{u_t : t>0\}$ will *not* satisfy (4.16.1), but these functions will be locally in L^2 in the sense that

$$\int_0^T \|u_t(x)\|^2\,dx < \infty$$

for every $T>0$. We will then use the results of section 3 to find a locally L^2 function $w:(0,\infty)\to\mathcal{C}$ with the property that for every $t>0$

$$u_t(x+t) = w(x+t) - w(x)$$

almost everywhere (dx) on the interval $(0,\infty)$. Such a function w can be subtracted from u_t

$$\phi_t(x) = \begin{cases} u_t(x) - w(x), & 0<x\le t \\ 0, & x>t \end{cases}$$

so that the modification $\{\phi_t : t>0\}$ satisfies both (4.16.1) and (4.16.2).

Lemma 4.19. *For each $s,t>0$, $\Gamma(s,t)(x)$ vanishes almost everywhere (dx) outside the interval $0<x<s+t$. Γ is a 2-cocycle in the sense that for every $r,s,t>0$ we have*

$$\Gamma(r+s,t) - \Gamma(r,s+t) - U_r\Gamma(s,t) + \Gamma(r,s) = 0. \tag{4.20}$$

proof. Since both e_se_t and e_{s+t} belong to $P(s+t)$, $\Gamma(s,t) = [e_se_t] - [e_{s+t}]$ belongs to

$$\overline{\text{span}}\{[x_1]-[x_2] : x_i \in P(s+t)\} = L^2((0,s+t);\mathcal{C})\}.$$

Moreover, since $e_{s+t} = e_se(s,s+t)$, we see that

$$\Gamma(s,t) = [e_se_t] - [e_se(s,s+t)] = U_s([e_t] - [e(s,s+t)])$$

belongs to the range of U_s and hence $\Gamma(s,t)$ must vanish almost everywhere on the interval $0<x\le s$.

To prove (4.20), notice that

$$U_r\Gamma(s,t) = U_r([e_se_t] - [e_{s+t}]) = [e_re_se_t] - [e_re_{s+t}],$$

and hence by the definition of Γ we have

$$\begin{aligned}\Gamma(r+s,t) &- \Gamma(r,s+t) - U_r\Gamma(s,t) \\ &= [e_{r+s}e_t] - [e_{r+s+t}] - [e_re_{s+t}] + [e_{r+s+t}] - [e_re_se_t] + [e_re_{s+t}] \\ &= [e_{r+s}e_t] - [e_re_se_t] = [e_{r+s}] - [e_re_s] = -\Gamma(r,s),\end{aligned}$$

as required □

Lemma 4.21. *For fixed $t > 0$ and $\xi \in L^2((0,\infty);\mathcal{C})$, the function*

$$s \mapsto \langle \Gamma(s,t), \xi \rangle$$

is Borel-measurable.

proof. Fix t. Noting that

$$\begin{aligned}\Gamma(s,t) &= [e_s e_t] - [e_{s+t}] = [e_s e_t] - [e_s e(s, s+t)] \\ &= U_s([e_t] - [e(s,s+t)]),\end{aligned}$$

we have

$$\langle \Gamma(s,t), \xi \rangle = \langle [e_t] - [e(s,s+t)], U_s^* \xi \rangle ,$$

for every $s, t > 0$. Since $s \mapsto U_s \xi$ is (metrically) continuous, it suffices to show that the function $s \in (0,\infty) \mapsto [e_t] - [e(s,s+t)]$ is weakly measurable; i.e., that

$$s \in (0,\infty) \mapsto \langle [e_t] - [e(s,s+t)], \eta \rangle$$

is a Borel function for every $\eta \in L^2((0,\infty);\mathcal{C})$. Since $L^2((0,T);\mathcal{C})$ is spanned by $\{[y_1] - [y_2] : y_i \in P(T)\}$ for every $T > 0$, this reduces to showing that for fixed T and $y_1, y_2 \in P(T)$,

$$s \in (0,\infty) \mapsto \langle [e_t] - [e(s,s+t)], [y_1] - [y_2] \rangle$$

is a Borel function. To see that, pick $u \in P(T-t)$. Then we have $[e_t] - [e(s,s+t)] = [e_t u] - [e(s,s+t)u]$, and if we set $x_1 = e_t u$ and $x_2 = e(s,s+t)u$ then $x_i \in P(T)$ and

$$\begin{aligned}\langle [e_t] - [e(s,s+t)], [y_1] - [y_2] \rangle =& \langle [x_1] - [x_2], [y_1] - [y_2] \rangle \\ =& \alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22},\end{aligned}$$

where $\alpha_{ij} = g(x_i, y_j)$. We have

$$\begin{aligned}\alpha_{1k} =& g(e_t u, y_k), \\ \alpha_{2k} =& g(e(s,s+t)u, y_k)\end{aligned}$$

for $k = 1, 2$. By the additivity property 2.3 we have

$$\begin{aligned}\alpha_{2k} =& g(e(s,s+t)u, y_k) = g(e(s,s+t)u, y(0,t)y_k(t,T)) \\ =& g(e(s,s+t), y_k(0,t)) + g(u, y_k(t,T)) + \psi(e(s,s+t), u) \\ &+ \overline{\psi(y_k(0,t), y_k(t,T))}.\end{aligned}$$

Noting that neither α_{11} nor α_{12} involves s and that the terms $\psi(e(s,s+t),u)$ cancel out of the difference $\alpha_{22} - \alpha_{21}$, we find that

$$\alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22} = -g(e(s,s+t), y_1(0,t)) + g(e(s,s+t), y_2(0,t)) + K$$

where K does not depend on s. The right side is a Borel function of s because of the measurability hypothesis on g $\square$

We define a family of functions $u_t : (0,\infty) \to \mathcal{C}$ as follows.

Lemma 4.22. *For every $s > 0$, the limit*

$$u_s(\lambda) = -\lim_{n\to\infty} \Gamma(s, n)(\lambda) \tag{4.22.1}$$

exists almost everywhere on $0 < \lambda < \infty$ and satisfies

$$\int_0^T \|u_s(\lambda)\|^2 \, d\lambda < \infty \tag{4.22.2}$$

for every $T > 0$. $\{u_s : s > 0\}$ is measurable in the sense that for every compactly supported $\xi \in L^2((0, \infty); \mathcal{C})$, the function

$$s \in (0, \infty) \mapsto \langle u_s, \xi\rangle$$

is Borel measurable. Putting $u_s(\lambda) = 0$ for $\lambda \le 0$ we have

$$\Gamma(s, t)(\lambda) = u_{s+t}(\lambda) - u_s(\lambda) - u_t(\lambda - s) \tag{4.22.3}$$

almost everywhere on $0 < \lambda < \infty$, for every $s, t > 0$.

Remark. Actually, the limit in (4.22.1) exists in a very strong sense. We will show that as t increases with s fixed, the restrictions

$$\Gamma(s, t) \restriction_{(0,T]}$$

stabilize as soon as t is larger than T. Once one knows this, the assertion (4.22.2) is an obvious consequence of the fact that each function $\Gamma(s, t)$ belongs to $L^2(0, \infty); \mathcal{C})$.

proof of Lemma 4.22. We first establish the coherence property described in the preceding remark. More precisely, we claim that for fixed $0 < s < T < t_1 < t_2$ one has

$$\Gamma(s, t_2) \restriction_{(0,T]} = \Gamma(s, t_1) \restriction_{(0,T]} . \tag{4.23}$$

To see that, consider the difference

$$\Gamma(s, t_2) - \Gamma(s, t_1) = [e_s e_{t_2}] - [e_{s+t_2}] - [e_s e_{t_1}] + [e_{s+t_1}].$$

Writing

$$\begin{aligned} [e_s e_{t_2}] - [e_s e_{t_1}] &= [e_s e_{T-s} e(T - s, t_2)] - [e_s e_{T-s} e(T - s, t_1)] \\ &= U_T([e(T - s, t_2)] - [e(T - s, t_1)]) \end{aligned}$$

and noting that

$$\begin{aligned} -[e_{s+t_2}] + [e_{s+t_1}] &= -[e_T e(T, t_2)] + [e_T e(T, t_1)] \\ &= -U_T([e(T, t_2)] - [e(T, t_1)]) \end{aligned}$$

we find that $\Gamma(s, t_2) - \Gamma(s, t_1)$ has the form $U_T \zeta$ for $\zeta \in L^2$ given by

$$\zeta = [e(T - s, t_2)] - [e(T - s, t_1)] - [e(T, t_2)] + [e(T, t_1)]$$

(4.23) follows because every function in the range of U_T vanishes a.e. on the interval $(0, T]$.

Thus (4.22.1) follows and by the preceding remark we also have (4.22.2). It is also clear that for every compactly supported function $\xi \in L^2$,

$$\langle u_s, \xi \rangle = \langle \Gamma(s, n), \xi \rangle$$

for sufficiently large $n = 1, 2, \dots$. Thus the measurability of $\{u_s\}$ follows as well.

Finally, the formula (4.22.3) follows after restricting all terms in the cocycle equation (4.16.2) to a finite unterval $0 < \lambda \leq T$ and taking the formal $\lim_{t \to \infty}$ to obtain

$$-u_{r+s} + u_r + U_r u_s + \Gamma(r, s+ = 0,$$

for every $r, s > 0$ □

We must now modify the family $\{u_s : s > 0\}$ in order to obtain a new family $\phi_s = u_s - w$ which has the additional property that $\phi_s(s)$ vanishes a.e. outside the interval $0 < \lambda \leq s$. This is accomplished as follows.

Notice that for $s, t > 0$,

$$u_{s+t}(\lambda) = u_s(\lambda) + u_t(\lambda - s)$$

almost everywhere on the interval $\lambda \geq s + t$. Indeed, this is immediate from the fact that

$$\Gamma(s, t)(\lambda) = u_{s+t}(\lambda) - u_s(\lambda) - u_t(\lambda - s)$$

and the fact that $\Gamma(s, t)$ vanishes outside the interval $0 < \lambda \leq s + t$. Thus if we define $v_t : (0, \infty) \to \mathcal{C}$ by

$$v_t(\lambda) = u_t(\lambda + t),$$

then $\{v_t : t > 0\}$ is a measurable family of $\mathcal{C}$-valued functions satisfying

$$\int_0^T \|v_t(\lambda)\|^2 \, d\lambda < \infty$$

for every $T, t > 0$, for which

$$v_{s+t}(\lambda) = v_s(\lambda) + v_t(\lambda + s)$$

almost everywhere $(d\lambda)$, for every $s, t > 0$. By Theorem 3.2, there is a Borel function

$$w : (0, \infty) \to \mathcal{C}$$

which is locally in L^2, such that for every $t > 0$ we have

$$v_t(\lambda) = w(\lambda + t) - w(\lambda)$$

almost everywhere on the interval $0 < \lambda < \infty$. Set $w(\lambda) = 0$ for $\lambda \leq 0$. It follows that

$$u_t(\lambda) - w(\lambda) + w(\lambda - t)$$

vanishes almost everywhere on the interval $t < \lambda < \infty$. Hence

$$\phi_t(\lambda) = u_t(\lambda) - w(\lambda) + w(\lambda - t)$$

satisfies both conditions

$$\phi_{s+t}(\lambda) - \phi_s(\lambda) - \phi_t(\lambda - s) = \Gamma(s,t)(\lambda)$$

a.e. on $0 < \lambda < \infty$ and

$$\phi_t(\lambda) = 0$$

almost everywhere on the interval $\lambda > t$. Notice that we can also define ϕ_t as follows

$$\phi_t(\lambda) = \begin{cases} u_t(\lambda) - w(\lambda), & 0 < \lambda \leq t \\ 0, & \lambda > t. \end{cases} \tag{4.24}$$

As in the remarks following the statement of Theorem 4.16, we can now define a fiber map $\log : P \to \mathcal{P}_{\mathcal{C}}$ by

$$\log(x) = [x] - [e_t] - \phi_t$$

for every $x \in P(t)$, and every $t > 0$, and this function satisfies $\log(P(t)) \subseteq \mathcal{P}_{\mathcal{C}}(t)$ and $\log(xy) = \log(x) \boxplus \log(y)$, for $x, y \in P$.

It remains to establish (4.3.1), and to exhibit a function $\rho : P \to \mathbb{C}$ which satisfies (4.3.2). ρ is defined as follows. If $x \in P(t)$ we put

$$\rho(x) = \langle [x] - [e_t], \phi_t\rangle + g(x, e_t) - \frac{1}{2}(g(e_t, e_t) + \|\phi_t\|^2).$$

To see that (4.3.2) is satisfied we choose $x_1, x_2 \in P(t)$ and use the definition of log to write

$$\begin{aligned}\langle \log(x_1), \log(x_2)\rangle &= \langle [x_1] - [e_t] - \phi_t, [x_2] - [e_t] - \phi_t\rangle \\ &= \langle [x_1] - [e_t], [x_2] - [e_t]\rangle - \langle [x_1] - [e_t], \phi_t\rangle \\ &\quad - \langle \phi_t, [x_2] - [e_t]\rangle - \|\phi_t\|^2.\end{aligned}$$

Noting that $\langle [x_1] - [e_t], [x_2] - [e_t]\rangle$ expands to

$$g(x_1, x_2) - g(x_1, e_t) - g(e_t, x_2) + g(e_t, e_t),$$

we obtain the required formula

$$\langle \log(x_1), \log(x_2)\rangle = g(x_1, x_2) + \rho(x_1) + \overline{\rho(x_2)}.$$

It remains to show that $\log(P(t))$ is a strongly spanning set in $\mathcal{P}_{\mathcal{C}}(t)$. To see that, fix $t > 0$, and let us write $L = \log(P(t))$. Notice that L carries no linear structure *a priori*, since the only algebraic property of the log function is its additivity

$$\log(xy) = \log(x) \boxplus \log(y).$$

Nevertheless, we will show that L is "almost convex".

For every $r > 0$, let

$$B_r = \{\xi \in \mathcal{P}_{\mathcal{C}}(t) : \|\xi\| \leq r\}$$

be the ball of radius r and let

$$K = \bigcup_{r>0} \overline{L \cap B_r}^{w}$$

where $\overline{L \cap B_r}^{w}$ denotes the closure of $L \cap B_r$ in the weak topology of the Hilbert space $\mathcal{P}_{\mathcal{C}}(t)$.

Lemma 4.26. *If $\xi, \eta \in K$ and θ is a dyadic rational in the unit interval, then $\theta\xi + (1-\theta)\eta \in K$.*

proof. It clearly suffices to prove that

$$\xi, \eta \in K \implies \frac{1}{2}(\xi + \eta) \in K.$$

We will show first that this is true in the special case where ξ, η have the form

$$\xi = \log(x), \quad \eta = \log(y)$$

with $x, y \in P(t)$. To this end we claim that there is a sequence $z_n \in P(t)$ with the properties

$$\|\log(z_n)\| \leq \|\log(x)\| + \|\log(y)\| \tag{4.27}$$

and which, in addition, satisfies

$$\lim_{n\to\infty} \langle \log(z_n), \zeta \rangle = \frac{1}{2} \langle \log(x), \zeta \rangle + \frac{1}{2} \langle \log(y), \zeta \rangle,$$

for all $\zeta \in \mathcal{P}_{\mathcal{C}}(t)$. Indeed, for every $n = 1, 2, \dots$, consider a dyadic partition of the interval $[0, t]$ as follows,

$$\{0 = t_0 < t_1 < \cdots < t_{2^n} = t\}$$

where $t_k = kt/2^n$, $0 \leq k \leq 2^n$. Using the propagators $\{x(r,s) : 0 \leq r < s \leq t\}$ and $\{y(r,s) : 0 \leq r < s \leq t\}$ for x and y we can define z_n as a product

$$z_n = x_1 y_2 x_3 y_4 \dots x_{2^n-1} y_{2^n}$$

where $x_k = x(t_{k-1}, t_k)$ and $y_k = y(t_{k-1}, t_k)$. Then because of the additivity property of log we have

$$\log(z_n) = \log(x_1) \boxplus \log(y_2) \boxplus \log(x_3) \boxplus \log(y_4) \boxplus \cdots \boxplus \log(x_{2^n-1}) \boxplus \log(y_{2^n}).$$

Letting $\mathcal{O}_n$ and $\mathcal{E}_n$ be the respective unions of the odd and even intervals,

$$\mathcal{O}_n = \bigcup_{k \text{ odd}} (t_{k-1}, t_k]$$
$$\mathcal{E}_n = \bigcup_{k \text{ even}} (t_{k-1}, t_k]$$

we can rewrite the previous formula for $\log(z_n)$ as follows

$$\log(z_n) = \log(x)\chi_{\mathcal{O}_n} + \log(y)\chi_{\mathcal{E}_n},$$

χ_S denoting the characteristic function of the set $S \subseteq [0, t]$. It follows that

$$\|\log(z_n)\| \leq \|\log(x)\| + \|\log(y)\|$$

Moreover, the arument of [2, p. 47] implies that for any function w in $L^1[0,t]$ we have

$$\lim_{n\to\infty}\int_{\mathcal{O}_n} w(x)\,dx = \frac{1}{2}\int_0^t w(x)\,dx \qquad \text{and}$$
$$\lim_{n\to\infty}\int_{\mathcal{E}_n} w(x)\,dx = \frac{1}{2}\int_0^t w(x)\,dx$$

(An equivalent assertion is that the sequence $\chi_{\mathcal{E}_n} \in L^\infty[0,t]$ converges to the constant function $1/2$ in the weak*-topology of $L^\infty[0,t]$). So if we fix $\zeta \in \mathcal{P}_{\mathcal{C}}(t) = L^2((0,t);\mathcal{C})$ then we have

$$\langle \log(z_n), \zeta\rangle = \langle \log(x)\cdot\chi_{\mathcal{O}_n} + \log(y)\chi_{\mathcal{E}_n}, \zeta\rangle = \int_{\mathcal{O}_n} w_1(\lambda)\,d\lambda + \int_{\mathcal{E}_n} w_2(x)\,dx,$$

where $w_1(\lambda) = \langle \log(x)(\lambda), \zeta(\lambda)\rangle$, $w_2(\lambda) = \langle \log(y)(\lambda), \zeta(\lambda)\rangle$. By the preceding remarks the right side tends in the limit to

$$\frac{1}{2}\int_0^t \langle \log(x)(\lambda), \zeta(\lambda)\rangle\; d\lambda + \frac{1}{2}\int_0^t \langle \log(y)(\lambda), \zeta(\lambda)\rangle\; d\lambda$$

as asserted.

Now let ξ, η be arbitrary elements of K. Since the sets $\overline{L\cap B_r}^w$ increase with r, we can assume that

$$\xi, \eta \in \overline{L\cap B_r}^w$$

for some $r > 0$. It follows that there are sequences $x_n, y_n \in P(t)$ satisfying $\|\log(x_n)\| \le r$, $\|\log(y_n)\| \le r$ and

$$\xi = \lim_{n\to\infty} \log(x_n),$$
$$\eta = \lim_{n\to\infty} \log(y_n)$$

weakly. Now for each fixed $n = 1, 2, \dots$ the preceding argument implies that

$$\frac{1}{2}\log(x_n) + \frac{1}{2}\log(y_n) \in \overline{L\cap B_{2r}}^w.$$

Since the set on the right is weakly closed, we may take the limit on n to obtain

$$\frac{1}{2}\xi + \frac{1}{2}\eta \in \overline{L\cap B_{2r}}^w,$$

as required $\square$

Remark 4.29. From Lemma 4.26 we immediately deduce: *the norm closure of*

$$K = \bigcup_{r>0} \overline{\log(P(t))\cap B_r}^w$$

is a convex subset of $\mathcal{P}_{\mathcal{C}}(t)$.

Lemma 4.30. *Set S be a convex subset of a Hilbert space H which spans H. Then S is a strongly spanning set.*

proof. If S contains 0 then the conclusion follows from [2 Proposition 6.12]. Thus we will obtain the more general result if we show that for every strongly spanning set $S_0 \subseteq H$ and every $\xi \in H$, $S_0 + \xi$ is also a strongly spanning set.

To see that, let $\zeta \in H \mapsto W_\zeta \in \mathcal{U}(\mathcal{B}(e^H))$ be the standard representation of the canonical commutation relations on the symmetric Fock space e^H; W_ζ is defined by requiring

$$W_\zeta : \exp(\eta) \mapsto e^{-\frac{1}{2}\|\eta\|^2 - \langle \eta, \zeta \rangle} \exp(\zeta + \eta),$$

for every $\eta \in H$. Each W_ζ is a unitary operator on e^H. Now let v be a vector in e^H such that $\langle v, \exp(\eta)\rangle = 0$ for every $\eta \in S_0 + \xi$. We have to show that $v = 0$. But for every $\eta_0 \in S_0$ we have

$$\langle v, W_\xi(\exp(\eta_0))\rangle = e^{-\frac{1}{2}\|\eta\|^2 - \langle \zeta, \eta\rangle} \langle v, \exp(\eta_0 + \xi)\rangle = 0$$

for every $\eta_0 \in S_0$, hence $W_\xi^* v$ is orthogonal to $\exp(S_0)$. Since $\exp(S_0)$ spans e^H we conclude that $W_\xi^* v = 0$, hence $v = 0$ □

We can now show that $L = \log(P(t))$ is a strongly spanning subset of $\mathcal{P}_\mathcal{C}(t)$. To see that, note first that the exponential map $\exp : \xi \mapsto \exp(\xi)$ is weakly continuous on *bounded* subsets of $\mathcal{P}_\mathcal{C}(t)$. Indeed, if $\{\xi_\alpha\}$ is a bounded net in $\mathcal{P}_\mathcal{C}(t)$ which converges weakly to ξ_∞, then for every $\eta \in \mathcal{P}_\mathcal{C}(t)$ we have

$$\begin{aligned} \lim_\alpha \langle \exp(\xi_\alpha), \exp(\eta)\rangle &= \lim_\alpha e^{\langle \xi_\alpha, \eta\rangle} = e^{\langle \xi_\infty, \eta\rangle} \\ &= \langle \exp(\xi_\infty), \exp(\eta)\rangle . \end{aligned}$$

Since $\exp(\xi_\alpha)$ is a bounded net in the symmetric Fock space over $\mathcal{P}_\mathcal{C}(t)$ and since the set of vectors $\{\exp(\eta) : \eta \in \mathcal{P}_\mathcal{C}(t)\}$ span this space, it follows that $\exp(\xi_\alpha)$ converges weakly to $\exp(\xi_\infty)$.

Now choose a vector v in the symmetric Fock space over $\mathcal{P}_\mathcal{C}(t)$ with the property that $\langle v, \exp(L)\rangle = \{0\}$. The preceding paragraph implies that v is orthogonal to the set of vectors $\exp(\overline{L \cap B_r}^w)$ for every $r > 0$, and taking the union over $r > 0$ we obtain $\langle v, \exp(K)\rangle = \{0\}$. Since the exponential map exp is *metrically* continuous on its entire domain $\mathcal{P}_\mathcal{C}(t)$, it follows that v is orthogonal to the set of vectors $\exp(\overline{K})$, $\overline{K}$ denoting the closure of K in the norm topology of $\mathcal{P}_\mathcal{C}(t)$. By remark 4.29 and Lemma 4.30, we conclude that $v = 0$.

That completes the proof of Theorem 4.3 □

Classification of multiplicative forms. We conclude this section with a discussion of how Theorem 4.3 gives a classification of multiplicative structures on path spaces. Let (P, g) be a metric path space and let $e^g : P^2 \to \mathbb{C}$ be its associated multiplicative form, defined on $P(t) \times P(t)$ for $t > 0$ by

$$x, y \in P(t) \mapsto e^{g(x,y)}.$$

e^g is a positive definite function on $P(t) \times P(t)$ and hence there is a Hilbert space $E(t)$ and a function $F_t : P(t) \to E(t)$ satisfying

$$E(t) = \overline{\text{span}}\{F_t(x) : x \in P(t)\}$$

$$\langle F_t(x), F_t(y)\rangle = e^{g(x,y)}, \quad x, y \in P(t)$$

Thus we have a family of Hilbert spaces $p : E \to (0, \infty)$

$$E = \{(t, \xi) : t > 0, \xi \in E(t)\}$$

with projection $p(t, \xi) = t$.

We define a binary operation $\xi, \eta \in E \mapsto \xi \cdot \eta \in E$ as follows. Since g is additive there is a function $\psi : P \times P \to \mathbb{C}$ such that for all $x_1, x_2 \in P(s)$, $y_1, y_2 \in P(t)$

$$g(x_1 y_1, x_2 y_2) = g(x_1, x_2) + g(y_1, y_2) + \psi(x_1, y_1) + \overline{\psi(x_2, y_2)}.$$

For $x \in P(s)$, $y \in P(t)$ we try to define the product $F_s(x) \cdot F_t(y)$ by

$$F_s(x) \cdot F_t(y) = e^{-\psi(x,y)} F_{s+t}(xy). \tag{4.31}$$

It follows that for $x_1, x_2 \in P(s)$, $y_1, y_2 \in P(t)$ we have

$$\begin{aligned} \langle F_s(x_1) \cdot F_t(y_1), F_s(x_2) \cdot F_t(y_2) \rangle &= e^{g(x_1 y_1, x_2 y_2) - \psi(x_1, y_1) - \overline{\psi(x_2, y_2)}} = e^{g(x_1, x_2) + g(y_1, y_2)} \\ &= \langle F_s(x_1), F_s(x_2) \rangle \langle F_t(y_1), F_t(y_2) \rangle . \end{aligned}$$

The latter formula implies that there is a unique unitary operator

$$W_{s,t} : E(s) \otimes E(t) \to E(s + t)$$

satisfying

$$W_{s,t}(F_s(x) \otimes F_t(y)) = F_s(x) \cdot F_t(y).$$

Thus we can define a bounded bilinear map $(\xi, \eta) \in E(s) \times E(t) \mapsto \xi \cdot \eta \in E(s + t)$ by way of

$$\xi \cdot \eta = W_{s,t}(\xi \otimes \eta)$$

and this mapping extends the operation (4.31).

To see that this operation on E is associative, it suffices to show that it is associative on generators, i.e.,

$$F_r(x) \cdot (F_s(y) \cdot F_t(z)) = (F_r(x) \cdot F_s(y)) \cdot F_t(z)$$

for all $x \in P(r), y \in P(s), z \in P(t)$. Using the definition (4.31), one observes that this will follow provided that ψ satisfies

$$\psi(x, y) + \psi(xy, z) = \psi(x, yz) + \psi(y, z). \tag{4.32}$$

In fact, the equation (4.32) can be arranged *a priori* from the definition of ψ (2.4). But it is easier at this point to invoke Theorem 4.3. The latter asserts that there is a complex-valued function ρ defined on P such that

$$\psi(x, y) = \rho(xy) - \rho(x) - \rho(y)$$

for all $x, y \in P$. Substituting this into (4.32) one finds (using associativity of the multiplication in P) that both sides of (4.32) reduce to

$$\rho(xyz) - \rho(x) - \rho(y) - \rho(z).$$

This proves associativity of the multiplication in E.

The preceding discussion implies that this multiplication acts like tensoring. Thus we have a product structure satisfying all the axioms of a product system except measurability requirements. Using Theorem 4.3, we can describe this product structure as follows.

Corollary 4.33. *Let (P, g) be a metric path space and let E be the product structure obtained from the positive definite functions*

$$(x, y) \in P(t) \times P(t) \mapsto e^{g(x,y)},$$

$t > 0$. *Assume that $E(t)$ is not one-dimensional for every t. Then E is isomorphic to the product structure of one of the standard product systems $E_1, E_2, \dots, E_\infty$.*

proof. By Theorem 4.3, there is a separable Hilbert space $\mathcal{C}$, a complex-valued function $\rho : P \to \mathbb{C}$ and a fiber map $\log : P \to \mathcal{P}_\mathcal{C}$ satisfying the conditions (4.3.1) and (4.3.2). In view of the remarks following Theorem 4.3, we may take the defect to be of the form

$$\psi(x, y) = \rho(xy) - \rho(x) - \rho(y).$$

Using the formula (4.31) we find that if we rescale $F_t : P(t) \to E(t)$ according to

$$g_t(x) = e^{-\rho(x)} F_t(x),$$

then the definition of multiplication in E simplifies to

$$G_s(x) \cdot G_t(y) = G_{s+t}(xy),$$

for $x \in P(s), y \in P(t)$, and all $s, t > 0$.

Moreover, formula (4.3.2) implies that for $x_1, x_2 \in P(t)$ we have

$$\langle G_t(x_1), G_t(x_2) \rangle = e^{g(x_1,x_2) - \rho(x_1) - \overline{\rho(x_2)}} = e^{\langle \log(x_1), \log(x_2) \rangle}.$$

Now consider the exponential map

$$\exp : L^2((0, \infty); \mathcal{C}) \to e^{L^2((0,\infty);\mathcal{C})}.$$

The latter formula asserts that

$$\langle G_t(x_1), G_t(x_2) \rangle = \langle \exp(\log(x_1)), \exp(\log(x_2)) \rangle$$

for all $x_1, x_2 \in P(t)$. This implies that we can define an isometry

$$W_t : E(t) \to e^{\mathcal{P}_\mathcal{C}(t)}$$

by way of

$$W_t(G_t(x)) = \exp(\log(x)),$$

for $x \in P(t)$. By (4.3.1), each W_t is a unitary operator. The total map

$$W : E \to e^{\mathcal{P}_\mathcal{C}}$$

is an isomorphism of families of Hilbert spaces.

It remains to verify that W preserves multiplication, i.e., that

$$W_{s+t}(\xi \cdot \eta) = W_s(\xi) W_t(\eta)$$

for every $\xi \in E(s)$, $\eta \in E(t)$. Recalling that the multiplication in $e^{\mathcal{P}_\mathcal{C}}$ is defined by

$$\exp(f)\exp(g) = \exp(f \boxplus g)$$

for $f \in \mathcal{P}_\mathcal{C}(s)$, $g \in \mathcal{P}_\mathcal{C}(t)$ we find that for all $x \in P(s)$, $y \in P(t)$,

$$\begin{aligned} W_{s+t}(G_s(x) \cdot G_t(y)) &= W_{s+t}(G_{s+t}(xy)) = \exp(\log(xy)) \\ &= \exp(\log(x) \boxplus \log(y)) = \exp(\log(x))\exp(\log(y)), \end{aligned}$$

and hence

$$W_{s+t}(G_s(x) \cdot G_t(y)) = W_s(G_s(x))W_t(G_t(y)).$$

The assertion follows from the bilinearity of multiplication and the fact that $E(r)$ is spanned by $G_r(P(r))$ for every $r > 0$

Finally, note that if the space $\mathcal{C}$ of "coordinates" is the trivial Hilbert space $\{0\}$ then $\mathcal{P}_\mathcal{C}(t) = L^2((0,t);\mathcal{C})$ is trivial as well and hence

$$e^{\mathcal{P}_\mathcal{C}(t)}$$

is one-dimensional for every $t > 0$. By virtue of the isomorphism $W : E \to e^{\mathcal{P}_\mathcal{C}}$, this has been ruled out in the hypothesis of Corollary 4.33. Thus $n = \dim(\mathcal{C})$ is a positive integer or $\aleph_0$. In this case, W implements an isomorphism of the product structure E onto the product structure of E_n $\square$

Part II. Continuous Tensor Products

Introduction to Part II.

Let $p : E \to (0, \infty)$ be a product system. Thus each fiber $E(t) = p^{-1}(t)$ is a separable Hilbert space and we are given an associative multiplication $(x, y) \in E \times E \mapsto xy \in E$ which acts like tensoring in the sense that for fixed $s, t > 0$,

$$(x, y) \in E(s) \times E(t) \mapsto xy \in E(s + t)$$

is a bilinear mapping with the properties

$$E(s + t) = \overline{\operatorname{span}} E(s)E(t) \tag{II.1}$$

$$\langle x_1 y_1, x_2 y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle , \tag{II.2}$$

for $x_i \in E(s)$, $y_i \in E(t)$. In addition, there are natural measurability axioms which we will not repeat here [2]. We will write $E = \{E(t) : t > 0\}$ instead of $p : E \to (0, \infty)$ when it is convenient.

A nonzero vector $x \in E(t)$ is called *decomposable* if for every $0 < s < t$ there are vectors $y \in E(s)$, $z \in E(t - s)$ for which

$$x = yz. \tag{II.3}$$

The set of decomposable vectors in $E(t)$ will be written $D(t)$.

There are product systems which contain no decomposable vectors at all. But if there is a $t_0 > 0$ for which $D(t_0) \neq \emptyset$, then $D(t) \neq \emptyset$ for every $t > 0$ and we clearly have

$$D(s + t) = D(s)D(t)$$

for every $s, t > 0$. This multiplicatively structured family of sets $\{D(t) : t > 0\}$ comes close to defining a path space, except that the factorizations of (II.3) are not unique. However, if $y_1, y_2 \in E(s)$ and $z_1, z_2 \in E(t-s)$ satisfy $y_1 z_1 = y_2 z_2$, then because of the identification of $E(t)$ with the tensor product $E(s) \otimes E(t-s)$ described in (II.1) and (II.2), we see that there must be a nonzero complex number λ such that

$$y_2 = \lambda y_1, \qquad z_2 = \lambda^{-1} z_1.$$

Thus we may obtain a path space structure by passing from each $D(t)$ to its associated projective space $\Delta(t)$.

More explictly, $\Delta(t)$ is obtained by identifying two vectors x_1, x_2 in $D(t)$ which are nonzero scalar multiples of each other. We consider $\Delta(t)$ to be a set with no additional structure. There is a natural projection $x \in D(t) \mapsto \dot{x} \in \Delta(t)$. Any complex function $f : D(t) \to \mathbb{C}$ which is homogeneous of degree zero in the sense that $f(\lambda x) = f(x)$ for all nonzero scalars λ can be promoted to a function $\dot{f} : \Delta(t) \to \mathbb{C}$ by way of $\dot{f}(\dot{x}) = f(x)$, $x \in D(t)$. In fact, it will be convenient to abuse notation slightly and identify functions on $\Delta(t)$ with homogenous functions defined on $D(t)$. The path space $p : \Delta \to (0, \infty)$ is defined by

$$\Delta = \{(t, \dot{x}) : t > 0, x \in D(t)\},$$

with projection $p(t, \dot{x}) = t$ and multiplication $(s, \dot{x})(t, \dot{y}) = (s + t, \dot{xy})$. We remind the reader that Δ, like any path space, is to be considered a fibered set with no additional structure beyond the multiplication it carries. Δ^2 will denote the fiber product

$$\Delta^2 = \{(t, \dot{x}, \dot{y}) : t > 0, \dot{x}, \dot{y} \in \Delta(t)\}.$$

For example, if for each $t > 0$ we are given a function $f_t : D(t) \to \mathbb{C}$ which satisfies $f_t(\lambda x) = f_t(x)$ for $x \in D(t)$ and $\lambda \neq 0$, then according to the abuses that have been agreed to we can define a function $\phi : \Delta \to \mathbb{C}$ by way of

$$\phi(t, x) = f_t(x), \qquad x \in D(t).$$

Of course, the inner product restricts to a positive definite function on every $D(t)$

$$(x, y) \in D(t) \times D(t) \mapsto \langle x, y \rangle \in \mathbb{C},$$

but this function cannot be promoted to one defined on $\Delta(t) \times \Delta(t)$. We will see in section 6 below that the inner product of any two vectors in $D(t)$ must be nonzero. Thus if we choose a fixed element $e \in D(t)$ then we may form the renormalized inner product

$$P_t(x, y) = \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle}$$

and the latter function can be promoted to a positive definite function on $\Delta(t) \times \Delta(t)$. If we choose a family $\{e_t \in D(t) : t > 0\}$ of decomposable vectors then we obtain a function

$$P : \Delta^2 \to \mathbb{C}$$

which restricts to a positive definite function on $\Delta(t) \times \Delta(t)$ for every $t > 0$. Of course, these renormalized versions of the inner product depend on the particular choice of $\{e_t : t > 0\}$.

The results of sections 5–9 below combine to show that it is possible to find additive forms $g : \Delta^2 \to \mathbb{C}$ for this path space which are logarithms of the inner product in the following sense

Theorem A. *Let $\{e_t \in D(t) : t > 0\}$ be a family of decomposable vectors which is left-coherent in the sense that for every s, t satisfying $0 < s < t$ there is a vector $e(s,t) \in D(t-s)$ such that*

$$e_t = e_s e(s,t).$$

Then there is an additive form $g : \Delta^2 \to \mathbb{C}$ (which will depend on e) such that for every $t > 0$ and every $x_1, x_2 \in D(t)$ we have

$$\langle x_1, x_2 \rangle = \langle x_1, e_t \rangle \, \overline{\langle x_2, e_t \rangle} e^{g(x_1, x_2)}.$$

In fact, we will show that $g : \Delta^2 \to \mathbb{C}$ is a "continuous" logarithm which "vanishes at zero"; moreover, it is uniquely determined by these requirements once $\{e_t\}$ is fixed. We emphasize that for typical choices of e the form g will have *nonzero* defect.

In sections 5–7, we establish certain continuity and nonvanishing properties of inner products of decomposable vectors. In section 8 we construct g as a "continuous" logarithm, and in section 9 we show that it has the required positivity properties.

5. Continuity of the modulus.

Let $E = \{E(t) : t > 0\}$ be a product system. Suppose that we are given vectors $x \in E(s)$, $y \in E(t)$ with $0 < s < t$. x is called a *left divisor* (resp. *right divisor*) of y if there is a vector $z \in E(t-s)$ such that $y = xz$ (resp. $y = zx$). Notice that in either case we have $\|x\| \cdot \|z\| = \|y\|$ and hence both x and z are nonzero whenever y is nonzero. Notice too that, while factorizations of the form II.3 are not unique, we do have both left and right cancellation laws. That is, if $y \in E(s)$ and $z_1, z_2 \in E(t)$, then $yz_1 = yz_2 \implies z_1 = z_2$, and $z_1 y = z_2 y \implies z_1 = z_2$.

Given $0 < T \leq \infty$, we say that a family of vectors $\{x_t \in E(t) : 0 < t < T\}$ is *left-coherent* (resp. *right-coherent*)if for every $0 < s_1 < s_2 < T$, x_{s_1} is a left (resp. right) divisor of x_{s_2}. Our analysis is based on the following continuity property of coherent families of vectors. The proof makes use of a central technical result from [3].

Theorem 5.1. *Let $\{x_t : 0 < t < T\}$ and $\{y_t : 0 < t < T\}$ be two left-coherent (resp. right-coherent) families of vectors satisfying $\|x_t\| = \|y_t\| = 1$ for all t. Then*

$$\lim_{t \to 0+} |\langle x_t, y_t \rangle| = 1.$$

Remarks. Notice that the quantity $|\langle x_t, y_t \rangle|$ does not exceed 1 and increases as t decreases to 0. Indeed, if $0 < s < t < T$ then in the left-coherent case we can write $x_t = x_s u$, $y_t = y_s v$ where u, v are unit vectors in $E(t-s)$, hence

$$|\langle x_t, y_t \rangle| = |\langle x_s u, y_s v \rangle| = |\langle x_s, y_s \rangle| \cdot |\langle u, v \rangle| \leq |\langle x_s, y_s \rangle| \leq 1.$$

It follows that the essential assertion of Theorem 5.1 is that

$$\sup_{0<t<T} |\langle x_t, y_t \rangle| = 1.$$

Notice too that we cannot draw any conclusion about continuity of the inner product itself. Indeed, if we start with $\{x_t\}$ as above and define y_t by $y_t = f(t)x_t$

where f is an arbitrary function from $(0, \infty)$ to the unit circle in the complex plane, then $< x_t, y_t >= f(t)$ can be pathological.

proof of Theorem 5.1. We will prove Theorem 5.1 for left-coherent families. With that in hand, the corresponding conclusion for right-coherent families follows from it by considering the product system E^o opposite to E (see [7,8]). Moreover, by passing to subfamilies if necessary, we may assume that T is finite and positive.

For every $s \in (0, T)$ we consider a projection $P_s \in \mathcal{B}(E(T))$ which is defined as follows. Since $\|x_s\| = 1$, the operator

$$L_{x_s} : z \in E(T - s) \to x_s z \in E_T$$

is an isometry whose range projection is given by

$$Ps = L_{x_s} L_{x_s}^* \in \mathcal{B}(E(T)).$$

The action of P_s on a product vector of the form

$$y = y_1 y_2, \qquad y_1 \in E(s), y_2 \in E(T - s)$$

is given by

$$P_s y = \langle y_1, x_s \rangle \, x_s y_2.$$

Note that if $0 < s_1 < s_2 < T$ then $P_{s_1} \geq P_{s_2}$, or equivalently that

$$x_{s_2} E(T - s_2) \subseteq x_{s_1} E(T - s_1). \tag{5.2}$$

Indeed, because of left-coherence x_{s_2} can be factored into a product $x_{s_2} = x_{s_1} z$ where $z \in E(s_2 - s_1)$. Hence $zE(T - s_2) \subseteq E(T - s_1)$, from which (5.2) is evident.

Therefore, as s decreases to 0 the projections P_s increase strongly to a limit projection P_0:

$$P_0 = \operatorname*{strong\,limit}_{s \to 0+} P_s.$$

We claim that $P_0 = \mathbf{1}$. Since P_0 is clearly a nonzero projection in $\mathcal{B}(E(T))$, it suffices to exhibit an irreducible $*$-algebra $\mathcal{A}$ of operators on $E(T)$ which commutes with P_0. We can exhibit such an algebra as follows.

For every $0 < s < T$ we can identify $E(T)$ with the tensor product $E(s) \otimes E(T - s)$ by associating a product $xy \in E(T)$, $x \in E(s)$, $y \in E(T - s)$ with $x \otimes y \in E(s) \otimes E(T - s)$. In this identification, P_s is associated with the projection

$$[x_s] \otimes \mathbf{1}_{E(T-s)},$$

$[x_s]$ denoting the rank-one projection $\xi \in E(s) \mapsto \langle \xi, x_s \rangle \, x_s \in E(s)$. Thus P_s commutes with the von Neumann algebra

$$\mathcal{A}_s = \mathbf{1}_s \otimes \mathcal{B}(E(T - s)).$$

As s decreases to 0 the operator algebras $\mathcal{A}_s$ increase, and hence P_0 commutes with the union

$$\mathcal{A} = \bigcup \mathcal{A}_s.$$

By Proposition 5.5 of [3], the union $\mathcal{A}$ is irreducible. Hence $P_0 = \mathbf{1}$.

It follows that

$$\lim_{s\to 0+} \|P_s\xi\| = \|\xi\|,$$

for every vector $\xi \in E(T)$. Taking $\xi = y_T$ and noting that for every $0 < s < T$, y factors into a product $y_T = y_s z_{T-s}$ where z_r is a unit vector in $E(r)$ for every $0 < r < T$, we have

$$P_s y_T = P_s(y_s z_{T-s}) = \langle y_s, x_s\rangle\, x_s z_{T-s}$$

and hence

$$1 = \|y_T\| = \lim_{s\to 0+} \|P_s y_T\| = \lim_{s\to 0+} |\,\langle y_s, x_s\rangle\,| \cdot \|x_s\| \cdot \|z_{T-s}\| = \lim_{s\to 0+} |\,\langle y_s, x_s\rangle\,|,$$

as required $\square$

With 5.1 in hand, we can now establish the following basic result on continuity of inner products.

Theorem 5.3. *Let $\{x_t : 0 < t < T\}$ and $\{y_t : 0 < t < T\}$ be two families of vectors which satisfy the hypotheses of Theorem 5.1. Then the function*

$$\phi(t) = |\,\langle x_t, y_t\rangle\,|$$

is continuous on the interval $0 < t < T$ and tends to 1 as $t \to 0+$.

proof. As in the proof of 5.1, it suffices to consider the case where both families are left-coherent, and where $\|x_t\| = \|y_t\| = 1$ for all t. Choose $t_0 \in (0, \infty)$. We will show that ϕ is both left continuous and right continuous at t_0.

right continuity. For every $\lambda \in (0, T - t_0)$ we may find u_λ, $v_\lambda \in E(\lambda)$ such that

$$x_{t_0+\lambda} = x_{t_0} u_\lambda,$$
$$y_{t_0+\lambda} = y_{t_0} v_\lambda.$$

Notice that $\|u_\lambda\| = \|v_\lambda\| = 1$. we claim that $\{u_\lambda : 0 < \lambda < T - t_0\}$ and $\{v_\lambda : 0 < \lambda < T - t_0\}$ are left-coherent families. Indeed, if $0 < \lambda_1 < \lambda_2 < T - t_0$ then since $\{x_t\}$ is left-coherent we may find $z \in E(\lambda_2 - \lambda_1)$ such that $x_{t_0+\lambda_2} = x_{t_0+\lambda_1} z$. Hence

$$x_{t_0} u_{\lambda_2} = x_{t_0+\lambda_2} = x_{t_0+\lambda_1} = x_{t_0} u_{\lambda_1} z.$$

From the left cancellation property we conclude that $u_{\lambda_2} = u_{\lambda_1} z$, proving that $\{u_\lambda\}$ is left-coherent. The same is true of $\{v_\lambda\}$.

Thus by Theorem 5.1 we have

$$\lim_{\lambda\to 0+} |\,\langle x_{t_0+\lambda}, y_{t_0+\lambda}\rangle\,| = \lim_{\lambda\to 0+} |\,\langle x_{t_0} u_\lambda, y_{t_0} v_\lambda\rangle\,| =$$
$$|\,\langle x_{t_0}, y_{t_0}\rangle\,| \lim_{\lambda\to 0+} |\,\langle u_\lambda, v_\lambda\rangle\,| = |\,\langle x_{t_0}, y_{t_0}\rangle\,|,$$

proving right continuity at λ

left continuity. Using left-coherence of $\{x_t\}$ and $\{y_t\}$, we may find vectors u_λ, $v_\lambda \in E_\lambda$ for every $\lambda \in (0, t_0)$ such that

$$x_{t_0} = x_{t_0-\lambda} u'_\lambda \tag{5.4}$$
$$y_{t_0} = y_{t_0-\lambda} v'_\lambda.$$

Notice that $\{u'_\lambda\}$ and $\{v'_\lambda\}$ are right-coherent families. The proof is similar to what was done above. For example, for $0 < \lambda_1 < \lambda_2 < t_0$ we may find $z' \in E(\lambda_2 - \lambda_1)$ such that

$$x_{t_0-\lambda_1} = x_{t_0-\lambda_2} z'.$$

Hence

$$x_{t_0-\lambda_2} u'_{\lambda_2} = x_{t_0-\lambda_1} u'_{\lambda_1} = x_{t_0-\lambda_2} z' u'_{\lambda_1},$$

and we obtain

$$u'_{\lambda_2} = z' u'_{\lambda_1}$$

after cancelling $x_{t_0-\lambda_2}$ from the left. The proof that $\{v'_\lambda\}$ is right-coherent is of course the same.

Using (5.4) we have

$$|\langle x_{t_0}, y_{t_0}\rangle| = |\langle x_{t_0-\lambda}, y_{t_0-\lambda}\rangle| \cdot |\langle u'_\lambda, v'_\lambda\rangle|$$

for every $0 < \lambda < t_0$. Because of Theorem 5.1,

$$\lim_{\lambda\to 0+} |\langle u'_\lambda, v'_\lambda\rangle| = 1,$$

hence

$$\lim_{\lambda\to 0+} |\langle x_{t_0-\lambda}, y_{t_0-\lambda}\rangle| = |\langle x_{t_0}, y_{t_0}\rangle|,$$

proving left continuity at t_0. The continuity of ϕ at $t = 0$ was established in Theorem 5.1 $\square$

6. Inner products of decomposable vectors.

Let $t > 0$ and let $x \in D(t)$ be a decomposable vector. Then for $0 < s < t$ there are vectors a_s, $b_s \in E(s)$ such that

$$x = a_s b_{t-s}. \tag{6.1}$$

Remark. Note that if $\|x\| = 1$, then $\|a_s\| \cdot \|b_{t-s}\| = 1$ and we may perform an obvious renormalization to achieve $\|a_s\| = \|b_s\| = 1$ for every s.

It is not obvious that each a_s and each b_s is a decomposable vector. The following lemma establishes this fact, and moreover it shows that $\{a_s : 0 < s < t\}$ (resp. $\{b_s : 0 < s < t\}$) is a left-decomposable (resp. right-decomposable) family.

Lemma 6.2. *Choose s_1, s_2 with $0 < s_1 < s_2 < t$, and suppose that $a_i \in E(s_i)$ and $b \in E(t - s_i)$ satisfy*

$$a_1 b_1 = a_2 b_2.$$

Then a_1 is a left-divisor of a_2 and b_2 is a right-divisor of b_1; i.e., there are vectors $c_1, c_2 \in E(s_2 - s_1)$ such that

$$a_2 = a_1 c_1$$
$$b_1 = c_2 b_2$$

proof. We may assume that $x = a_1b_1 = a_2b_2$ satisfies $\|x\| = 1$, and thus after an obvious renormalization we may also assume $\|a_i\| = \|b_i\| = 1$.

We require an enhanced version of the right cancellation law. Note that if z is any element of E, say $z \in E(\lambda)$ for $\lambda > 0$, then the right multiplication map $R_z : u \in E \mapsto uz \in E$ restricts to a bounded linear map on fiber spaces, carrying $E(\mu)$ to $E(\mu + \lambda)$, and thus has a fiber adjoint. Let

$$R_z^* : \{E(\mu) : \mu > \lambda\} \to E$$

be the total map defined by these adjoints. We claim that that R_z^* commutes with any left multiplication operator $L_a : u \mapsto au$ in the sense that L_a commutes with the restriction of R_z^* to any fiber space in any case in which the formulas make sense. That is, if $a \in E(\lambda)$ and $u \in E(\mu)$ with $\mu > \lambda$, then we have

$$aR_z^*(u) = R_z^*(au). \tag{6.3}$$

To see this, simply note that since $E(\mu)$ is spanned by $E(\mu - \lambda)E(\lambda)$, it suffices to verify that (6.3) is valid for vectors u of the form u_1u_2 with $u_1 \in E(\mu - \lambda)$ and $u_2 \in E(\lambda)$. In that case the left side of 6.3 is

$$aR_z^*(u_1u_2) = a(u_1 \langle u_2, z\rangle)$$

while the right side is

$$R_z^*(a(u_1u_2)) = R_z^*((au_1)u_2) = au_1 \langle u_2, z\rangle,$$

as asserted.

We apply these remarks to the proof of Lemma 6.2 as follows. Write

$$a_2 = a_2\|b_2\|^2 = R_{b_2}^*(a_2b_2) = R_{b_2}^*(a_1b_1) = a_1R_{b_2}^*(b_1),$$

and therefore we can take $c_1 = R_{b_2}^*(b_1) \in E(s_2 - s_1)$.

The other decomposition $b_1 = c_2b_2$ follows from this by considering the product system E^o opposite to E. Indeed, if we interpret the equation $a_1b_1 = a_2b_2$ in E^o, it becomes $b_1a_1 = b_2a_2$. By what was proved above, there is an element $c_2 \in E^o(s_2 - s_1)$ for which $b_1 = b_2c_2$, and if we interpret the latter in E we obtain $b_1 = c_2b_2$. $\square$

Remark 6.4. We may conclude that every vector $x \in D(t)$ can be associated with a propagator $\{x(r, s) \in D(s - r) : 0 \le r < s \le t\}$ which satisfies $x(0, t) = x$. Indeed, for each $0 < s < t$ we can find a nonzero left-divisor $x_s \in E(s)$ for x. Set $x_t = x$. By Lemma 6.2, $\{x_s : 0 < s \le t\}$ is a left-coherent family with $x_s \in D(s)$ for every s. Because of the left cancellation law, we can therefore define a propagator $\{x(r, s) : 0 \le r < s \le t\}$ by setting

$$x_s = x_rx(r, s)$$

for $0 < r < s \le t$, and by setting $x(0, s) = x_s$.

Theorem 6.5. *For any $t > 0$ and any two vectors $x, y \in D(t)$ we have $\langle x, y\rangle \neq 0$.*

proof. There is clearly no loss if we assume that $\|x\| = \|y\| = 1$. By the preceding remarks, we can find propagators $\{x(r,s) : 0 \leq r < s \leq t\}$ and $\{y(r,s) : 0 \leq r < s \leq t\}$ for x and y with the property that $x(0,t) = x$ and $y(0,t) = y$. By an obvious renormalization, we can also assume that $\|x(r,s)\| = \|y(r,s)\| = 1$ for every $0 \leq r < s \leq t$.

Notice that for every $0 < r \leq t$ we have

$$\lim_{\lambda\to 0+} |\,\langle x(r-\lambda, r), y(r-\lambda, r)\rangle\,| = 1. \tag{6.6}$$

Indeed, this follows by applying 5.3 to the right-coherent normalized sections $a_\lambda = x(r-\lambda, r)$ and $b_\lambda = y(r-\lambda, r)$, $0 < \lambda < r$.

Now the function $f : [0,t] \to \mathbb{R}$ defined by

$$f(s) = \begin{cases} |\,\langle x(0,s), y(0,s)\rangle\,|, & 0 < s \leq t, \\ 1, & s = 0 \end{cases}$$

is continuous, by Theorem 5.3. We claim that f is never zero. For if there were an $r \in [0,t]$ for which $f(r) = 0$, then there is a smallest one r_0, and we must have $0 < r_0 \leq t$. But for every $s \in (0, r_0)$ we can write

$$\begin{aligned} f(r_0) =&|\,\langle x(0,r_0), y(0,r_0)\rangle\,| = |\,\langle x(0,s)x(s,r_0), y(0,s)y(s,r_0)\rangle\,| \\ =&f(s)|\,\langle x(s,r_0), y(s,r_0)\rangle\,|. \end{aligned}$$

$f(s)$ is nonzero for every $s < r_0$, and because of (6.6) the term $|\,\langle x(s,r_0), y(s,r_0)\rangle\,|$ is nonzero when s is sufficiently close to r_0. This contradicts the fact that $f(r_0)$ was supposed to be zero. Taking $s = t$ we find that $|\,\langle x, y\rangle\,| = f(t) \neq 0$ ☐

7. Continuity and normalization of the inner product.

We will write D for the set of all left-coherent decomposable sections. Thus D consists of all sections

$$t \in (0,\infty) \mapsto x_t \in E(t)$$

which are left-coherent and for which x_t is never zero. It is possible, of course, that $D = \emptyset$. But if D is not empty then we are interested in establishing continuity of the inner product function

$$t \in (0,\infty) \mapsto \langle x_t, y_t\rangle\,, \tag{7.1}$$

defined by two elements $x, y \in D$. In this section we will show that if one normalizes the elements of D appropriately then inner products of the form (7.1) are continuous.

Remarks. Note that in general, nothing can be said about continuity of the inner products (7.1) (see the remarks following Theorem 5.1).

Notice too that, even though unique factorization fails in the multiplicative family $\{D(t) : t > 0\}$ we still have a left cancellation law, and this implies that there is a bijective correspondence between left-coherent sections and propagators. For example, if $x \in D(t)$ and we find a left-coherent family $\{x_s : 0 < s \leq t\}$ for which

$x_t = x$ then we may define a unique propagator $\{x(r,s) \in D(t-s) : 0 \le r < s \le t\}$ by

$$x_s = x_r x(r,s)$$

for $0 < r < s$ (by the left cancellation property), and where for $r = 0$ we put $x(0,s) = x_s$. The left-coherent family is recovered from its propagator via $x_s = x(0,s)$, $0 < s \le t$.

The normalization in D is done as follows. Choose an arbitrary element $e \in D$ satisfying $\|e_t\| = 1$ for every $t > 0$; e will be fixed throughout the remainder of this section.

Definition 7.3. *D^e is the set of all $x \in D$ satisfying $\langle x_t, e_t\rangle = 1$ for every $t > 0$.*

Lemma 7.4. *For every $x \in D^e$, the norm $\|x_t\|$ is a continuous nondecreasing function of t with*

$$\lim_{t\to 0+} \|x_t\| = 1.$$

In particular, we have $\|x_t\| \ge 1$ for every $t > 0$.

proof. Let $u_t = \|x_t\|^{-1} x_t$. Then both $\{e_t\}$ and $\{u_t\}$ are left-coherent families of *unit* vectors. So by Theorems 5.1 and 5.3 we may conclude that $|\langle u_t, e_t\rangle|$ is continuous in t over the interval $(0,\infty)$ and satisfies

$$\lim_{t\to 0} |\langle u_t, e_t\rangle| = 1.$$

Since

$$|\langle u_t, e_t\rangle| = \|x_t\|^{-1} |\langle x_t, e_t\rangle| = \|x_t\|^{-1},$$

the continuity assertion follows.

To see that $\|x_t\|$ increases with t, choose $0 < s < t$. By left-coherence of $\{x_t\}$ and $\{e_t\}$ we can write

$$e_t = e_s u,$$
$$x_t = x_s v,$$

where $u = e(s,t)$, and $v = x(s,t)$ belong to $E(t-s)$. Note that u must be a unit vector because $\|e(s,t)\| = \|e_t\|/\|e_s\| = 1$. Notice too that $\|v\| \ge 1$. Indeed, since $\langle e_r, x_r\rangle = 1$ for all r we can write

$$\langle u, v\rangle = \langle e_s, x_s\rangle \langle u, v\rangle = \langle e_s u, x_s v\rangle = \langle e_t, x_t\rangle = 1,$$

so by the Schwarz inequality

$$1 = |\langle u, v\rangle| \le \|u\| \cdot \|v\| = \|v\|.$$

It follows that

$$\|x_s\| \le \|x_s\| \cdot \|v\| = \|x_s v\| = \|x_t\|$$

as asserted □

Our principal result on the continuity of inner products is the following.

Theorem 7.5. *Let $x, y \in D^e$. Then the inner product $\langle x_t, y_t\rangle$ is continuous and nonzero on $0 < t < \infty$, and satisfies*

$$\lim_{t\to 0+} \langle x_t, y_t\rangle = 1.$$

proof. We will deduce Theorem 7.5 from the following inequality. For every s, t, T satisfying $0 < s < t \le T < \infty$ we claim that

$$|\,\langle x_s, y_s\rangle - \langle x_t, y_t\rangle\,| \le \|x_T\| \cdot \|y_T\| \sqrt{(\|x_t\|^2 - \|x_s\|^2)(\|y_t\|^2 - \|y_s\|^2)}. \tag{7.6}$$

To prove (7.6), we write

$$x_t = x_s u$$
$$y_t = y_s v$$
$$e_t = e_s f$$

where $u = x(s,t), v = y(s,t), f = e(s,t) \in E(t-s)$ and $\|f\| = 1$. Notice that

$$\langle u, f\rangle = \langle v, f\rangle = 1. \tag{7.7}$$

Indeed, since $\langle x_s, e_s\rangle = 1$ we have

$$\langle u, f\rangle = \langle x_s, e_s\rangle \langle u, f\rangle = \langle x_s u, e_s f\rangle = \langle x_t, e_t\rangle = 1,$$

and similarly $\langle v, f\rangle = 1$. We can therefore estimate the quantity $\langle u, v\rangle - 1$ as follows:

$$|\,\langle u, v\rangle - 1| = |\,\langle u - f, v - f\rangle\,| \le \|u - f\| \cdot \|v - f\|.$$

By Lemma 7.3 we have $\|x_s\| \ge 1$. Hence we may use $\langle u, f\rangle = 1$ again to obtain

$$\|u - f\|^2 = \|u\|^2 - 1 = \frac{\|x_t\|^2}{\|x_s\|^2} - 1 = \|x_s\|^{-2}(\|x_t\|^2 - \|x_s\|^2) \le \|x_t\|^2 - \|x_s\|^2.$$

Similarly,

$$\|v - f\| \le \sqrt{\|y_s\|^2 - \|y_s\|^2}.$$

Thus

$$|\,\langle u, v\rangle - 1| \le \sqrt{(\|x_t\|^2 - \|x_s\|^2)(\|y_t\|^2 - \|y_s\|^2)}.$$

The inequality (7.6) follows after multiplying the preceding inequality through by $|\,\langle x_s, y_s\rangle\,|$, noting that

$$\begin{aligned} |\,\langle x_s, y_s\rangle\,| \cdot |\,\langle u, v\rangle - 1| &= |\,\langle x_s, y_s\rangle \langle u, v\rangle - \langle x_s, y_s\rangle\,| \\ &= |\,\langle x_t, y_t\rangle - \langle x_s, y_s\rangle\,|, \end{aligned}$$

and using the Schwarzenegger inequality and Lemma 7.4 to estimate the factor $|\,\langle x_s, y_s\rangle\,|$ on the right by way of

$$|\,\langle x_s, y_s\rangle\,| \le \|x_s\| \cdot \|y_s\| \le \|x_T\| \cdot \|y_T\|.$$

This establishes (7.6).

Now Lemma 7.4 implies that $\|x_t\|^2$ and $\|y_t\|^2$ are continuous increasing functions tending to 1 as $t \to 0+$, and from (7.6) we immediately conclude that $\langle x_t, y_t\rangle$ is continuous on $(0, \infty)$. If we allow s to tend to 0 in (7.6) and use $\lim_{t\to 0+} \|x_s\| = 1$ from Lemma 7.4 the result is

$$|\,\langle x_t, y_t\rangle - 1| \le \|x_T\| \cdot \|y_T\| \sqrt{(\|x_t\|^2 - 1)(\|y_t\|^2 - 1)},$$

from which we deduce

$$\lim_{t\to 0+} \langle x_t, y_t\rangle = 1.$$

That establishes continuity on the closed interval $[0, \infty)$ $\square$

8. Continuous logarithms.

Fix $t > 0$. We have seen above that if x, y are two decomposable vectors in $E(t)$ then the inner product $\langle x, y\rangle$ is not zero. Thus one might attempt to define a logarithm function $(x, y) \in D(t) \mapsto L(t; x, y) \in \mathbb{C}$ with the property that

$$e^{L(t;x,y)} = \langle x, y\rangle,$$

in such a way that the logarithms fit together consistently for different values of t. We will show that this is in fact possible, provided that one is careful to define the logarithm so as to remove as much ambiguity as possible. In section 9 we will show that L is conditionally positive definite.

Let $\Delta = \{\Delta(t) : t > 0\}$ be the path space obtained from $\{D(t) : t > 0\}$ as in the introduction to Part II above. Despite the fact that Δ and Δ^2 are lifeless sets, there is a useful notion of continuity for complex functions defined on them. We will say that $\phi : \Delta \to \mathbb{C}$ is continuous if, for every left-coherent section $t \in (0, \infty) \mapsto u_t \in D(t)$, the function $f(t) = \phi(t; u_t)$ is continuous on the interval $(0, \infty)$, and the limit

$$f(0) = \lim_{t\to 0+} f(t)$$

exists. For functions $\psi : \Delta^2 \to \mathbb{C}$, continuity means that for any pair of sections $u, v \in D$, the function $g(t) = \psi(t; u_t, v_t)$ is continuous for positive t and extends continuously to $[0, \infty)$. We will say that ϕ (resp. ψ) *vanishes at the origin* if $f(0) = 0$ (resp. $g(t) = 0$) for all choices of u (resp. u, v).

Remark 8.1. Notice that certain normalized inner products give rise to continuous functions $F : \Delta^2 \to \mathbb{C}$. For example, with e as above put

$$F(t; x, y) = \frac{\langle x, y\rangle}{\langle x, e_t\rangle \langle e_t, y\rangle},$$

for $t > 0$, $x, y \in D(t)$. To see that F is continuous choose $u, v \in D$ and put $u'_t = \langle u_t, e_t\rangle^{-1} u_t$ and $v'_t = \langle v_t, e_t\rangle^{-1} v_t$. Then we have

$$F(t; u_t, v_t) = \langle u'_t, v'_t\rangle$$

because of the homogeniety of F. Theorem 7.5 implies that the right side is continuous in t and tends to 1 as $t \to 0+$. It follows that $F : \Delta^2 \to \mathbb{C}$ is a continuous function. Needless to say, F depends on e.

Theorem 8.2. *Let $e \in D$ satisfy $\|e_t\| = 1$ for every $t > 0$. Then there is a unique continuous function $L^e : \Delta^2 \to \mathbb{C}$ which vanishes at the origin and satisfies*

$$e^{L^e(t;x,y)} = \frac{\langle x, y\rangle}{\langle x, e_t\rangle \langle e_t, y\rangle}, \tag{8.2a}$$

for every $t > 0$, $x, y \in D(t)$.

If $f \in D$ satisfies $\|f_t\| = 1$ for every $t > 0$ and $L^f : \Delta^2 \to \mathbb{C}$ is the corresponding logarithm, then there is a continuous function $\phi : \Delta \to \mathbb{C}$ which vanishes at 0 and satisfies

$$L^f(t; x, y) = L^e(t; x, y) + \phi(t; x) + \overline{\phi(t; y)} \tag{8.2b}$$

for all $t > 0$, $x, y \in D(t)$.

Remark 8.3. The function $L^e : \Delta^2 \to \mathbb{C}$ is called the *e-logarithm* of the inner product on E. Notice that we can use L^e to define a logarithm of the non-normalized inner product in the following way. For every $t > 0$ and every $x \in D(t)$ let $f(t; x)$ be a complex number such that

$$e^{f(t;x)} = \langle x, e_t \rangle .$$

The function f need have no regularity properties whatsoever, and may even be non-measurable. Nevertheless, once we settle on f then we can define

$$L(t; x, y) = L^e(t; x, y) + f(t; x) + \overline{f}(t; y)$$

and this new L will satisfy

$$e^{L(t;x,y)} = \langle x, y \rangle , \qquad x, y \in D(t), t > 0.$$

We will see below that $L^e(t; \cdot, \cdot)$ is a positive definite function on $D(t) \times D(t)$, and hence $L(t; \cdot, \cdot)$ is a *conditionally positive definite logarithm of the inner product* $\langle \cdot, \cdot \rangle : D(t) \times D(t) \to \mathbb{C}$. Now the construction of a Hilbert space from a conditionally positive definite function has the property that $L(t; \cdot, \cdot)$ and $L^e(t; \cdot, \cdot)$ determine the same Hilbert space (see section 7). It follows that the function f has no effect on the invariantly defined Hilbert spaces that concern us. Moreover, for the same reason *(8.2b) implies that these Hilbert spaces will also be independent of the particular choice of normalized section* $e \in D$.

proof of Theorem 8.2. For uniqueness, notice that if $L^e, M^e : \Delta^2 \to \mathbb{C}$ both satisfy the conditions associated with (8.2a), then

$$\phi(t; x, y) = L^e(t; x, y) - M^e(t; x, y)$$

is a continuous complex-valued function on Δ^2 which vanishes at the origin and satisfies

$$e^{\phi(t;x,y)} = 1$$

identically. To see that $\phi = 0$ choose $t > 0$ and $x, y \in D(t)$. Let $u, v \in D$ be left-coherent sections such that u_t and v_t are, respectively, scalar multiples of x and y (see Theorem 10.1). The function $s \in (0.\infty) \to \phi(t; u_s, v_s) \in \mathbb{C}$ is continuous, vanishes as $s \to 0+$ and satisfies

$$e^{\phi(s;u_s,v_s)} = 1$$

for all $s > 0$. Hence $\phi(s; u_s, v_s) = 0$ for all s. By homogeneity, $\phi(t; x, y) = \phi(t; u_t, v_t) = 0$.

For existence, fix $t > 0$, $x, y \in D(t)$. We define $L^e(t; x, y)$ as follows. Again, we find sections $u, v \in D$ such that $u_t = \lambda x$, $v_t = \mu y$, with $\lambda\mu \neq 0$. Now the function

$$s \in (0, \infty) \mapsto \frac{\langle u_s, v_s \rangle}{\ldots}$$

is continuous, never 0, and tends to 1 as $s \to 0+$ (see Remark 8.1). Thus there is a unique continuous function $l \in C[0,\infty)$ satisfying the conditions $l(0) = 0$ and

$$e^{l(s)} = \frac{\langle u_s, v_s\rangle}{\langle u_s, e_s\rangle \langle e_s, v_s\rangle}, \qquad s > 0. \tag{8.4}$$

We define $L^e(t;x,y) = l(t)$.

To see that $L^e(t;x,y)$ is well-defined, choose another pair $u', v' \in D$ so that $u'_t = \lambda' x$ and $v'_t = \mu' y$ with $\lambda'\mu' \neq 0$. Choose $l' \in C[0,\infty)$ with $l'(0) = 0$ so that (8.4) is satisfied with u', v' replacing u, v. We have to show that $l'(t) = l(t)$. But for $0 < s \leq t$ the uniqueness of factorizations of the two vectors x, y implies that there are nonzero complex numbers α_s, β_s so that $u'_s = \alpha_s u_s$, $v'_s = \beta_s v_s$ for $0 < s \leq t$. It follows that the right side of (8.4) is unaffected by passing from u, v to u', v'. Hence $l' = l$ and finally $l'(t) = l(t)$.

Notice that the continuity of L^e follows from its definition. Indeed, for u, v and l related by (8.4) in the definition of $L^e(t;x,y)$, we must also have

$$L^e(s; u_s, v_s) = l(s)$$

for every $0 < s \leq t$. In particular, the function $s \in (0,t] \mapsto L^e(s;u_s,v_s)$ is continuous and tends to 0 as $s \to 0+$. Since t, x, y can be chosen arbitrarily, the sections $u, v \in D$ are also arbitrary. It follows that $L^e : \Delta^2 \to \mathbb{C}$ is continuous and vanishes at 0.

To prove (8.2b), pick $f \in D$ so that $\|f_t\| = 1$ for every t, and consider the function $\Phi : \Delta \to \mathbb{C}$ defined by

$$\Phi(t;x) = \frac{|\langle e_t, f_t\rangle| \langle x, e_t\rangle}{\langle f_t, e_t\rangle \langle x, f_t\rangle}, \qquad t > 0, \quad x, y \in D(t).$$

We claim that for every $u \in D$, $\Phi(t, u_t)$ is continuous in $t \in (0,\infty)$ and tends to 1 as $t \to 0+$. Indeed, putting $u' = \langle u_t, f_t\rangle^{-1} u_t$ and $e'_t = \langle e_t, f_t\rangle^{-1} e_t$, then e' and u' are elements of D satisfying $\langle e'_t, f_t\rangle = \langle u'_t, f_t\rangle = 1$, and

$$\Phi(t; u_t) = |\langle e_t, f_t\rangle| \langle u'_t, e'_t\rangle$$

for $t > 0$. The claim follows because both $|\langle e_t, f_t\rangle|$ and $\langle u'_t, e'_t\rangle$ are continuous in t and tend to 1 as $t \to 0+$ by Theorem 7.5.

To define ϕ we proceed as we did in defining the function L^e above. Fix $t > 0$, $x \in D(t)$, and choose $u \in D$ so that $u_t = \lambda x$ for some complex number $\lambda \neq 0$. By the preceding paragraph there is a unique continuous function $l \in C[0,t]$ so that $l(0) = 0$ and

$$e^{l(s)} = \Phi(s; u_s), \qquad 0 < s \leq t.$$

Put $\phi(t;x) = l(t)$. One shows that ϕ is well-defined and continuous as one did for L^e.

Finally, since both $L^f(t;x,y) - L^e(t;x,y)$ and $\phi(t;x) + \overline{\phi(t;y)}$ define continuous functions on Δ^2 which vanish at 0, (8.2b) will follow if we show that

(8.5) $e^{L^f(t;x,y) - L^e(t;x,y)} \qquad e^{\phi(t;x)+\overline{\phi(t;y)}}$

The left side of (8.5) is the quotient of

$$e^{L^f(t;x,y)} = \frac{\langle x, y\rangle}{\langle x, f_t\rangle \langle f_t, y\rangle}$$

by the quantity

$$e^{L^e(t;x,y)} = \frac{\langle x, y\rangle}{\langle x, e_t\rangle \langle e_t, y\rangle}.$$

Thus the $\langle x, y\rangle$ terms cancel out and the left side of (8.5) reduces to

$$\frac{\langle x, e_t\rangle \langle e_t, y\rangle}{\langle x, f_t\rangle \langle f_t, y\rangle}. \tag{8.6}$$

Similarly, the right side is the product of

$$e^{\phi(t;x)} = \frac{|\langle e_t, f_t\rangle|\langle x, e_t\rangle}{\langle f_t, e_t\rangle \langle x_t, f_t\rangle}$$

with the quantity

$$\overline{e^{\phi(t;y)}} = \frac{|\langle e_t, f_t\rangle|\langle e_t, y\rangle}{\langle e_t, f_t\rangle \langle f_t, y\rangle}.$$

After performing the indicated multiplication all terms involving $\langle e_t, f_t\rangle$ cancel and the result agrees with (8.6) □

Remark 8.7. We remark that for each e, the e-logarithm is self-adjoint in the sense that

$$\overline{L^e(t; x, y)} = L^e(t; y, x), \qquad t > 0, \quad x, y \in D(t).$$

To see that, simply note that the function $F : \Delta^2 \to \mathbb{C}$ defined by

$$F(t; x, y) = \overline{L^e(t; y, x)}$$

has all of the defining properties of an e-logarithm, and hence $F = L^e$ by uniqueness.

In the applications of part III, we will need to know that the function L^e defines an addtitive form on Δ^2. The following establishes this fact.

Proposition 8.8. *Fix $s > 0$. Then there is a continuous function $\psi_s : \Delta \to \mathbb{C}$, vanishing at 0, such that for all $x_1, x_2 \in \Delta(s)$, all $t > 0$ and all $y_1, y_2 \in \Delta(t)$ we have*

$$L^e(s+t; x_1y_1, x_2y_2) - L^e(s; x_1, x_2) - L^e(t; y_1, y_2) = \psi_s(t; y_1) + \overline{\psi_s(t; y_2)}. \tag{8.9}$$

proof. Fix $s > 0$. We claim that there is a continuous function $\psi_s : \Delta \to \mathbb{C}$ which vanishes at 0 and satisfies

$$e^{\psi_s(t;y)} = \frac{|\langle e(s, s+t), e_t\rangle|\langle y, e_t\rangle}{\langle y, e(s, s+t)\rangle \langle e(s, s+t), y\rangle}$$

for all $y \in D(t)$, $t > 0$. In order to define ψ_s, fix $t > 0$ and $y_0 \in D(t)$. Choose a left-decomposable section $y \in D$ such that y_t is a scalar multiple of y_0. The function is continuous and tends to 1 as $t \to 0+$ by Theorem 2.3. Similarly, by Remark 6.1, the function

$$t \in (0, \infty) \to \frac{\langle y_t, e_t\rangle}{\langle y_t, e(s, s+t)\rangle \langle e(s, s+t), y_t\rangle}$$

has the same properties. Thus

$$t \in (0, \infty) \to \frac{|\langle e(s, s+t), e_t\rangle| \langle y_t, e_t\rangle}{\langle y_t, e(s, s+t)\rangle \langle e(s, s+t), y_t\rangle}$$

is a continuous function which tends to 1 as $t \to 0+$. It follows that there is a unique continuous function $l : [0, \infty) \to \mathbb{C}$ such that $l(0) = 0$ and

$$e^{l(t)} = \frac{|\langle e(s, s+t), e_t\rangle| \langle y_t, e_t\rangle}{\langle y_t, e(s, s+t)\rangle \langle e(s, s+t), y_t\rangle}$$

for $t > 0$. We define $\psi_s(t; y_0) = l(t)$.

$\psi_s(t; \cdot)$ is a homogeneous function of degree 0 on $D(t)$ and hence we may consider ψ_s to be a function defined on Δ. ψ_s is continuous because of the way it was defined.

It remains to show that ψ_s satisfies (8.9). For that, it suffices to show that for any pair of left-decomposable sections $t \to y_t, y'_t \in D(t)$ we have

$$L^e(s+t; x_1 y_t, x_2 y'_t) - L^e(s; x_1, x_2) - L^e(t; y_t, y'_t) = \psi_s(t; y_t) + \overline{\psi_s(t; y'_t)}$$

for every $t > 0$. To see this, let $L(t)$ and $R(t)$ be the left and right sides of the preceding formula. $R(t)$ is continuous on $(0, \infty)$ and tend to 0 as $t \to 0+$ by definition of ψ_s, and we claim that $L(\cdot)$ has these two properties as well. Indeed, $L(t)$ is continuous for positive t because of the continuity of L^e. To see that $L(t) \to 0$ as $t \to 0+$, consider the left-coherent sections u, u' defined by

$$u_r = \begin{cases} (x_1)_r, & 0 < r \le s \\ (x_1)_s y_{r-s}, & r > s \end{cases}, \qquad u'_r = \begin{cases} (x_2)_r, & 0 < r \le s \\ (x_2)_s y'_{r-s}, & r > s \end{cases}.$$

By continuity of $L^e(r; u_r, u'_r)$ at $r = s$ we obtain

$$\begin{aligned} \lim_{t\to 0+} L(t) &= \lim_{t\to 0+} (L^e(s+t; u_{s+t}, u'_{s+t}) - L^e(s; x_1, x_2) - L^e(t; y_t, y'_t)) \\ &= L^e(s; u_s, u'_s) - L^e(s; x_1, x_2) = 0, \end{aligned}$$

as asserted.

Since both $L(t)$ and $R(t)$ are continuous on $(0, \infty)$ and tend to 0 as $t \to 0+$, it suffices to show that

$$e^{L(t)} = e^{R(t)}, \qquad \text{for } t > 0.$$

But

$$\begin{aligned} e^{R(t)} &= \frac{|\langle e(s, s+t), e_t\rangle| \langle y_t, e_t\rangle}{\langle y_t, e(s, s+t)\rangle \langle e(s, s+t), e_t\rangle} \cdot \left(\overline{\frac{|\langle e(s, s+t), e_t\rangle| \langle y'_t, e_t\rangle}{\langle y'_t, e(s, s+t)\rangle \langle e(s, s+t), e_t\rangle}}\right) \\ &= \frac{\langle y_t, e_t\rangle \langle e_t, y'_t\rangle}{\cdots} \end{aligned}$$

while

$$e^{L(t)} = \frac{\langle x_1, y_t, x_2 y_t' \rangle}{\langle x_1 y_t, e_{s+t} \rangle \langle e_{s+t}, x_2 y_t' \rangle} \cdot \frac{\langle x_1, e_s \rangle \langle e_s, y_2 \rangle}{\langle x_1, x_2 \rangle} \cdot \frac{\langle y_1, e_t \rangle \langle e_t, y_2 \rangle}{\langle y_1, y_2 \rangle}.$$

Using the formulas

$$\begin{aligned} &\langle x_1 y_t, x_2 y_t' \rangle = \langle x_1, x_2 \rangle \langle y_t, y_t' \rangle \\ &\langle x_1 y_t, e_{s+t} \rangle = \langle x_1 y_t, e_s e(s, s+t) \rangle = \langle x_1, e_s \rangle \langle y_t, e(s, s+t) \rangle \\ &\langle e_{s+t}, x_2 y_t' \rangle = \langle e_s, x_2 \rangle \langle e(s, s+t), y_t' \rangle \end{aligned}$$

and performing the obvious cancellations, we obtain

$$e^{L(t)} = \frac{\langle y_t, e_t \rangle \langle e_t, y_t' \rangle}{\langle y_t, e(s, s+t) \rangle \langle e(s, s+t, y_t' \rangle} = e^{R(t)},$$

as required $\square$

9. Infinite divisibility of the inner product.

We have indicated in remark 8.3 how to find functions L of the form

$$L(t; x, y) = L^e(t; x, y) + f(t, x) + \overline{f(t; y)}$$

that are logarithms of the inner product restricted to decomposable vectors:

$$e^{L(t;x,y)} = \langle x, y \rangle, \qquad t > 0, \quad x, y \in D(t).$$

It is essential for the constructions of Part III that such an L should have the property that for fixed $t > 0$, it defines a conditionally positive definite function on $D(t) \times D(t)$. According to the remarks at the beginning of section 2, it would be enough to exhibit a sequence of positive definite functions $\Phi_n : D(t) \times D(t) \to \mathbb{C}$ (depending on t) such that

$$\Phi_n(x, y)^n = \langle x, y \rangle, \qquad x, y \in D(t)$$

for every $n = 1, 2, \ldots$. Unfortunately, there are no natural candidates for the positive definite functions Φ_n. Thus we will have to establish the conditional positive definiteness of $L(t; \cdot, \cdot)$ directly, by making use of the structure of the product system itself. Actually, we will prove somewhat more than we require.

Theorem 9.2. *Let $e \in D$ satisfy $\|e_t\| = 1, t > 0$. Then for every $t > 0$ the function*

$$(x, y) \in D(t) \times D(t) \mapsto L^e(t; x, y)$$

is positive definite.

The proof of Theorem 9.2 will occupy the remainder of this section. Note that once 9.2 has been proved, one can immediately deduce

Corollary 9.3. *For every $t > 0$, the inner product of E restricts to an infinitely divisible positive definite function on $D(t) \times D(t)$.*

proof of Theorem 9.2. Let e be an element of D satisfying $\|e_t\| = 1$ for all $t > 0$, which will be fixed throughout the remainder of this section. We will define a function

$$P^e : \Delta^2 \to \mathbb{C}$$

with the property that each function

$$(x, y) \in D(t) \times D(t) \mapsto P^e(t; x, y)$$

is *obviously* positive definite, and which is also a continuous e-logarithm that vanishes 0. The conclusion $L^e = P^e$ will then follow by the uniqueness assertion of Theorem 8.2 and hence we obtain 9.2.

Turning now to the proof, fix $t > 0$ and choose $x, y \in D(t)$ with $\langle x, e_t \rangle = \langle y, e_t \rangle = 1$. Because of the normalization of x and y there are left-coherent families $\{x_s : 0 < s \le t\}$ and $\{y_s : 0 < s \le t\}$ satisfying $\langle x_s, e_s \rangle = \langle y_s, e_s \rangle = 1$ for all s with the property that $x_t = x$ and $y_t = y$. Moreover, the two families are uniquely determined by these conditions because of the uniqueness of factorizations. If $I = (a, b]$ is a subinterval of $(0, t]$ having positive length then we will write x_I (resp. y_I) for the value of the propagator $x(a, b)$ (resp. $y(a, b)$). Finally, if $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_m = t\}$ and $\mathcal{Q} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ are two partitions of $(0, t]$, we will write $\mathcal{P} \le \mathcal{Q}$ for the usual ordering $\mathcal{P} \subseteq \mathcal{Q}$. Thus we can define a net of complex numbers on the increasing directed set of partitions by

$$B_{\mathcal{P}}(t; x, y) = \sum_{I \in \mathcal{P}} (\langle x_I, y_I \rangle - 1). \tag{9.4}$$

Proposition 9.5. *For each $t > 0$ and every partition $\mathcal{P}$ of $(0, t]$, $B_{\mathcal{P}}(t; \cdot, \cdot)$ is a positive definite function whose associated Hilbert space is separable. The net is decreasing in the sense that for $\mathcal{P} \le \mathcal{Q}$, $B_{\mathcal{P}}(t; \cdot, \cdot) - B_{\mathcal{Q}}(t; \cdot, \cdot)$ is a positive definite function on $D(t) \times D(t)$.*

proof. Let $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. To see that $B_{\mathcal{P}}(t; \cdot, \cdot)$ is positive definite, choose $x, y \in D(t)$ and let $\{x_s : 0 < s \le t\}$ and $\{y_s : 0 < s \le t\}$ be the unique families defined above. It will be convenient to write $x_k = x(t_{k-1}, t_k)$, $y_k = y(t_{k-1}, t_k)$ and $e_k = e(t_{k-1}, t_k)$. Noting that $\langle x_k, e_k \rangle = \langle y_k, e_k \rangle = 1$ we have $\langle x_k, y_k \rangle - 1 = \langle x_k - e_k, y_k - e_k \rangle$ and thus we can write

$$B_{\mathcal{P}}(t; x, y) = \sum_{k=1}^{n} (\langle x_k, y_k \rangle - 1) = \sum_{k=1}^{n} \langle x_k - e_k, y_k - e_k \rangle .$$

Notice that $x_k - e_k$ and $y_k - e_k$ belong to the Hilbert space $E(t_k - t_{k-1})$ for every $k = 1, 2, \dots, n$. Thus we can define a vector-valued function

$$F_{\mathcal{P}} : D(t) \to E(t_1) \oplus E(t_2 - t_1) \oplus \cdots \oplus E(t_n - t_{n-1})$$

by way of

$$F_{\mathcal{P}}(x) = (x_1 - e_1, x_2 - e_2, \dots, x_n - e_n)$$

The preceding formula for $B_{\mathcal{P}}(t;x,y)$ now becomes

$$B_{\mathcal{P}}(t;x,y) = \langle F_{\mathcal{P}}(x), F_{\mathcal{P}}(y)\rangle .$$

This formula shows that $B_{\mathcal{P}}(t;\cdot,\cdot)$ is a positive definite function. Moreover, since the range of the function $F_{\mathcal{P}}$ is contained in a separable Hilbert space, it follows that the Hilbert space associated with $B_{\mathcal{P}}(t;\cdot,\cdot)$ is separable.

It remains to show that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions satisfying $\mathcal{P}_1 \leq \mathcal{P}_2$ then $B_{\mathcal{P}_1}(t;\cdot,\cdot) - B_{\mathcal{P}_2}(t;\cdot,\cdot)$ is a positive definite function. Now since the partial order of positive definite functions defined by ($B_1 \leq B_2 \iff B_2 - B_1$ is positive definite) is transitive and since $\mathcal{P}_2$ is obtained from $\mathcal{P}_1$ by a sequence of steps in which one refines a single interval at every step, we can reduce to the case in which $\mathcal{P}_1 = \{0 = s_0 < s_1 < \cdots < s_m = t\}$ and $\mathcal{P}_2$ is obtained by adding a single point c to $\mathcal{P}_1$, where $s_{k-1} < c < s_k$ for some $k = 1, 2, \ldots, m$. In this case the difference $\Delta = B_{\mathcal{P}_1} - B_{\mathcal{P}_2}$ is given by

$$\begin{aligned}\Delta(x,y) &= \langle x(s_{k-1},s_k), y(s_{k-1},s_k)\rangle - 1\\ &\quad - (\langle x(s_{k-1},c), y(s_{k-1},c)\rangle + \langle x(c,s_k), y(c,s_k)\rangle - 2)\\ &= \langle x(s_{k-1},s_k), y(s_{k-1},s_k)\rangle - \langle x(s_{k-1},c), y(s_{k-1},c)\rangle - \langle x(c,s_k), y(c,s_k)\rangle\\ &\quad + 1.\end{aligned} \tag{9.6}$$

If we write $x_1 = x(s_{k-1},c)$, $x_2 = x(c,s_k)$, $y_1 = y(s_{k-1},c)$, $y_2 = y(c,s_k)$ then the right side of (9.6) can be rewritten as follows

$$\begin{aligned}&\langle x_1x_2, y_1y_2\rangle - \langle x_1,y_1\rangle - \langle x_2,y_2\rangle + 1\\ &\qquad = \langle x_1,y_1\rangle\langle x_2,y_2\rangle - \langle x_1,y_1\rangle - \langle x_2,y_2\rangle + 1\\ &\qquad = (\langle x_1,y_1\rangle - 1)(\langle x_2,y_2\rangle - 1) = \langle x_1 - e_1, y_1 - e_1\rangle\langle x_2 - e_2, y_2 - e_2\rangle\\ &\qquad = \langle (x_1 - e_1)(x_2 - e_2), (y_1 - e_1)(y_2 - e_2)\rangle\end{aligned}$$

where $e_1 = e(s_{k-1},c)$, $e_2 = e(c,s_k)$, and where the inner product in the last term on the right is taken in the Hilbert space $E((s_k - c) + (c - s_{k-1}) = E(s_k - s_{k-1})$. The last term clearly defines a positive definite function of x and y □

The conditions of Proposition 9.5 imply that the pointwise limit $\lim_{\mathcal{P}} B_{\mathcal{P}}(t;x,y)$ exists. This is a consequence of the following elementary result.

Lemma 9.7. *Let I be a directed set and let $\{P_\alpha : \alpha \in I\}$ be a net of positive definite functions on a set X which is decreasing in the sense that $\alpha \leq \beta$ implies that $P_\alpha - P_\beta$ is positive definite. Then*

$$P_\infty(x,y) = \lim_\alpha P_\alpha(x,y)$$

exists for every $x, y \in X$ and P_∞ is a positive definite function. If the Hilbert space associated with some P_α is separable then so is the Hilbert space associated with P_∞.

proof. Since a pointwise limit of positive definite functions is obviously positive definite, we merely show that the above limit exists and defines a separable Hilbert space.

Fix two elements $x, y \in X$. For every $\alpha \in I$ consider the 2×2 complex matrix

$$A_\alpha = \begin{pmatrix} P_\alpha(x,x) & P_\alpha(x,y) \\ P_\alpha(y,x) & P_\alpha(y,y) \end{pmatrix}.$$

We may consider $\{A_\alpha : \alpha \in D\}$ as a net of self adjoint operators on the two dimensional Hilbert space $\mathbb{C}^2$. We have $A_\alpha \geq 0$ for every α because P_α is positive definite, and $\alpha \leq \beta \implies A_\beta \leq A_\alpha$ because the net P_α is decreasing. Hence the net of operators A_α must converge in the weak operator topology to a positive operator

$$A = \lim_\alpha A_\alpha.$$

Considering A as a 2×2 matrix, the element λ in the 12 position satisfies

$$\lambda = \lim_\alpha P_\alpha(x,y),$$

establishing the existence of the required limit.

For separability, notice that there are Hilbert spaces H_α, H_∞ and functions $F_\alpha : X \to H_\alpha$, $F_\infty : X \to H_\infty$ with the property

$$P_\alpha(x,y) = \langle F_\alpha(x), F_\alpha(y)\rangle,$$
$$P_\infty(x,y) = \langle F_\infty(x), F_\infty(y)\rangle,$$

and where we may also assume H_α (resp. H_∞) is spanned by $F_\alpha(X)$ (resp. $F_\infty(X)$). By hypothesis, we can find α so that H_α is separable. Since $P_\alpha - P_\infty$ is positive definite it follows that there is a unique contraction $T : H_\alpha \to H_\infty$ having the property $T(F_\alpha(x)) = F_\infty(x)$ for every $x \in X$. Thus T maps H_α onto a dense subspace of H_∞. Since H_α is separable we conclude that H_∞ is separable as well. □

By 9.5 and 9.7, we may define a positive definite function $P_\infty(t; \cdot, \cdot)$ on $D(t) \times D(t)$ by

$$B_\infty(t; x, y) = \lim_{\mathcal{P}} B_{\mathcal{P}}(t; x, y).$$

Finally, we define $P^e : \Delta^2 \to \mathbb{C}$ by

$$P^e(t; x, y) = B_\infty(t; \langle x, e_t\rangle^{-1} x, \langle y, e_t\rangle^{-1} y).$$

It remains to show that P^e is an e-logarithm, i.e., that it is continuous, vanishes at the origin, and exponentiates correctly.

We deal first with continuity and vanishing at 0. Choose $u, v \in D$ so that $\langle u_t, e_t\rangle = \langle v_t, e_t\rangle = 1$ for every t. We have to show that the function

$$t \in (0,\infty) \mapsto P^e(t; u_t, v_t) = B_\infty(t; u_t, v_t)$$

is continuous on $(0, \infty)$ and tends to 0 as $t \to 0+$. This will follow from the following two estimates.

Proposition 9.8. *If* $0 < s < t$ *and* u, v *are as above, then*

$$\begin{aligned} |P^e(s; u_s, v_s)| \leq& (\|u_s\|^2 - 1)(\|v_s\|^2 - 1), \\ |P^e(t; u_t, v_t) - P^e(s; u_s, v_s)| \leq& (\|u_t\|^2 - \|u_s\|^2)(\|v_t\|^2 - \|v_s\|^2). \end{aligned}$$

proof. Consider the first of the two inequalities. Because of the fact that

$$P^e(s; u_s, v_s) = \lim_{\mathcal{P}} B_{\mathcal{P}}(s; u_s, v_s),$$

it suffices to show that for every partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_m = s\}$ of the interval $(0, s]$ we have

$$|B_{\mathcal{P}}(s; u_s, v_s)|^2 \leq (\|u_s\|^2 - 1)(\|v_s\|^2 - 1). \tag{9.9}$$

For that, let us write $u^k = u(s_{k-1}, s_k)$, $v^k = v(s_{k-1}, s_k)$, $e^k = e(s_{k-1}, s_k)$ for $k = 1, 2, \dots, m$. Because of the normalizations $\langle u^k, e^k \rangle = \langle v^k, e^k \rangle = 1$ we have

$$\langle u^k, v^k \rangle - 1 = \langle u^k - e^k, v^k - e^k \rangle$$

and hence

$$\begin{aligned} |B_{\mathcal{P}}(s; u_s, v_s)|^2 &= |\sum_{k=1}^m \langle u^k - e^k, v^k - e^k \rangle|^2 \leq (\sum_{k-1}^m \|u^k - e^k\| \cdot \|v^k - e^k\|)^2 \\ &\leq \sum_{k=1}^m \|u^k - e^k\|^2 \sum_{k=1}^m \|v^k - e^k\|^2. \end{aligned}$$

Now we can write

$$\|u^k - e^k\|^2 = \|u^k\|^2 - 1 = \|u(s_{k-1}, s_k)\|^2 - 1.$$

If $k = 1$ this is just $\|u_{t_1}\|^2 - 1$, and if $k > 1$ it becomes

$$\begin{aligned} \frac{\|u_{s_k}\|^2}{\|u_{s_{k-1}}\|^2} - 1 &= \|u_{s_{k-1}}\|^{-2}(\|u_{s_k}\|^2 - \|u_{s_{k-1}}\|^2) \\ &\leq \|u_{s_k}\|^2 - \|u_{s_{k-1}}\|^2. \end{aligned}$$

Thus we can estimate $\sum \|u^k - e^k\|^2$ using a telescoping series to obtain

$$\sum_{k=1}^m \|u^k - e^k\|^2 \leq \|u_{s_m}\|^2 - 1 = \|u_s\|^2 - 1.$$

Similarly,

$$\sum_{k=1}^m \|v^k - e^k\|^2 \leq \|v_{s_m}\|^2 - 1 = \|v_s\|^2 - 1,$$

and the first of the two inequalities follows.

The proof of the second is similar, and we merely indicate the changes. It suffices to show that for any partition $\mathcal{P}$ of $(0, t]$ which contains s, we have

$$|B_{\mathcal{P}}(t; u_t, v_t) - B_{\mathcal{P}\cap(0,s]}(s; u_s, v_s)| \leq (\|u_t\|^2 - \|u_s\|^2)(\|v_t\|^2 - \|v_s\|^2).$$

The desired inequality will follow by taking the limit on $\mathcal{P}$. Suppose that

$$\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_m = s = t_0 < t_2 < \cdots < t_n = t\}.$$

Then if we write out the formula for $B_{\mathcal{P}}(t; u_t, v_t)$ we find that

$$B_{\mathcal{P}}(t; u_t, v_t) = B_{\mathcal{P}\cap(0,s]}(s; u_s, v_s) + \sum_{l=1}^{n}(\langle u(t_{l-1}, t_l), v(t_{l-1}, t_l)\rangle - 1).$$

Thus we have to show that

$$|\sum_{l=1}^{n}(\langle u(t_{l-1}, t_l), v(t_{l-1}, t_l)\rangle - 1)|^2 \leq (\|u_t\|^2 - \|u_s\|^2)(\|v_t\|^2 - \|v_s\|^2).$$

But if we write $u^l = u(t_{l-1}, t_l)$, $v^l = v(t_{l-1}, t_l)$, $e^l = e(t_{l-1}, t_l)$, then we have

$$\left\langle u^l, v^l \right\rangle - 1 = \left\langle u^l - e^l, v^l - e^l \right\rangle$$

and as in the proof of the first inequality it suffices to show that

$$\sum_{l=1}^{n} \|u^l - e^l\|^2 \leq \|u_t\|^2 - \|u_s\|^2,$$

with a similar estimate for u replaced with v. But noting that

$$\{s = t_0 < t_1 < \cdots < t_n = t\}$$

is a partition of the interval $(s, t]$, we can make similar estimates as those made in the preceding argument to estimate the sum

$$\sum_{l=1}^{n} \|u^l - e^l\|^2 = \sum_{l=1}^{n}(\|u(t_{l-1}, t_l)\|^2 - 1)$$

with a telescoping series whose sum is $\|u_t\|^2 - \|u_s\|^2$ $\square$

From the inequalities of Proposition 9.8 and Lemma 7.4, we immediately conclude that $P^e : \Delta^2 \to \mathbb{C}$ is continuous and vanishes at the origin.

Thus, to show that P^e is an e-logarithm (and therefore coincides with L^e) it remains only to show that for $t > 0$ and $x, y \in D(t)$ we have

$$e^{P^e(t;x,y)} = \frac{\langle x, y\rangle}{\langle x, e_t\rangle \langle e_t, y\rangle}.$$

Since both sides are homogeneous functions of degree zero in x and y, it suffices to prove the formula for x, y normalized so that $\langle x, e_t\rangle = \langle y, e_t\rangle = 1$. That is, we must prove that

$$e^{B_\infty(t;x,y)} = \langle x, y\rangle\,, \tag{9.10}$$

for all $x, y \in D(t)$ satisfying $\langle x, e_t\rangle = \langle y, e_t\rangle = 1$. We will deduce (9.10) from the following lemma, which may be considered a generalization of the familiar formula

$$\lim_{n\to\infty}(1 + z/n)^n = e^z.$$

l^1 (resp. l^2) will denote the space of all sequences of complex numbers $z = (z(1), z(2), \dots)$ which are summable (resp. square summable). The norm of $z \in l^2$ is denoted $\|z\|$.

Lemma 9.11. *Let I be a directed set and let $\{z_\alpha : \alpha \in I\}$ be a net of sequences in $l^1 \cap l^2$ satisfying*

$$\lim_\alpha \|z_\alpha\|_2 = 0, \text{ and} \tag{9.11a}$$

$$\lim_\alpha \sum_{k=1}^\infty z_\alpha(k) = \zeta \in \mathbb{C}. \tag{9.11b}$$

Then for every $\alpha \in I$ the infinite product $\prod_{k=1}^\infty (1 + z_\alpha(k))$ converges absolutely and we have

$$\lim_\alpha \prod_{k=1}^\infty (1 + z_\alpha(k)) = e^\zeta.$$

Remarks. Notice that since $\sum_{k=1}^\infty |z_\alpha(k)|$ converges for every $\alpha \in I$, every infinite product $p_\alpha = \prod_{k=1}^\infty (1 + z_\alpha(k))$ converges absolutely as well.

Note too that that Lemma 9.11 reduces to the familiar formula

$$\lim_{n\to\infty} (1 + \zeta/n)^n = e^\zeta,$$

by taking $I = \{1, 2, \dots\}$ and

$$z_n = (\underbrace{\zeta/n, \zeta/n, \dots, \zeta/n,}_{n \text{ times}} 0, \dots).$$

In this case (9.11a) follows from the fact that $\|z_n\|_2 = |\zeta|/n^{1/2} \to 0$ as $n \to \infty$, and (9.11b) is an identity for every n.

proof of Lemma 9.11. Let log denote the principal branch of the complex logarithm on the region $\{1 + z : |z| < 1\}$. Then for sufficiently large α we have

$$\sup_{k\geq 1} |z_\alpha(k)| \leq \|z_\alpha\|_2 < 1,$$

and for such an α $\log(1 + z_\alpha(k))$ is defined for every $k = 1, 2, \dots$. We will show that the series $\sum_k \log(1 + z_\alpha(k))$ is absolutely convergent for large α and, in fact,

$$\lim_\alpha \sum_{k=1}^\infty \log(1 + z_\alpha(k)) = \zeta. \tag{9.12}$$

The required conclusion follows after exponentiating (9.12).

Now since

$$\lim_{z\to 0} \frac{\log(1 + z) - z}{z^2} = 1/2 < 1,$$

we can find $\epsilon > 0$ so that

$$|\log(1 + z) - z| < |z|^2$$

for all z with $|z| < \epsilon$. If α is large enough that $\|z_\alpha\|_2 < \epsilon$ then we have

$$\sum^\infty |\log(1 + z_\alpha(k)) - z_\alpha| \leq \sum^\infty |z_\alpha(k)|^2 < \epsilon^2.$$

In particular, $\sum_k |\log(1+z_\alpha(k))| < \infty$ because $z_\alpha \in l^1$. Now we obtain (9.12) by noting that for large α,

$$\begin{aligned}|\sum_k \log(1+z_\alpha(k)) - \zeta| \leq &\sum_k |\log(1+z_\alpha(k)) - z_\alpha(k)| + |\sum_k z_\alpha(k) - \zeta| \\ \leq &\epsilon^2 + |\sum_k z_\alpha(k) - \zeta|,\end{aligned}$$

and using the hypothesis (9.11b) to estimate the second term on the right. □

We apply Lemma 9.11 to the directed set I of all partitions $\mathcal{P}$ of the interval $(0,t]$ as follows. For every

$$\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$$

we have from (9.8)

$$B_{\mathcal{P}}(t;x,y) = \sum_{k=1}^n z_{\mathcal{P}}(k)$$

where $z_{\mathcal{P}}(k) = \langle x(t_{k-1},t_k), y(t_{k-1},t_k)\rangle - 1$. Now for $0 \leq a < b \leq t$ we have

$$\begin{aligned}|\langle x(a,b), y(a,b)\rangle - 1| &= |\langle x(a,b) - e(a,b), y(a,b) - e(a,b)\rangle| \\ &\leq \|x(a,b) - e(a,b)\| \cdot \|y(a,b) - e(a,b)\|.\end{aligned}$$

Let $\phi, \psi : [0,t] \to \mathbb{R}$ be the functions $\phi(0) = \psi(0) = 1$, $\phi(s) = \|x_s\|^2$, $\psi(s) = \|y_s\|^2$ for $0 < s \leq 1$. Then ϕ and ψ are continuous monotone increasing, and an estimate like those in the proof of Theorem 9.8 shows that

$$\|x(a,b) - e(a,b)\|^2 = \|x(a,b)\|^2 - 1 \leq \phi(b) - \phi(a)$$

and

$$\|y(a,b) - e(a,b)\|^2 = \|y(a,b)\|^2 - 1 \leq \psi(b) - \psi(a).$$

Thus we can estimate $|z_{\mathcal{P}}(k)|$ as follows. Setting

$$\epsilon(\alpha) = \sup_{|t-s|\leq\alpha} |\phi(t) - \phi(s)|$$

for positive α, we have

$$|z_{\mathcal{P}}(k)|^2 \leq (\phi(t_k) - \phi(t_{k-1}))(\psi(t_k) - \psi(t_{k-1})) \leq \epsilon(|\mathcal{P}|)(\psi(t_k) - \psi(t_{k-1})),$$

$|\mathcal{P}|$ denotinging the norm of $\mathcal{P}$. It follows that

$$\sum_{k=1}^n |z_{\mathcal{P}}(k)|^2 \leq \epsilon(|\mathcal{P}|)(\phi(t) - 1).$$

Since $\lim_{\mathcal{P}} \epsilon(|\mathcal{P}|) = 0$ because $\lim_{\mathcal{P}} |\mathcal{P}| = 0$ and ϕ is uniformly continuous, we conclude that $\lim_{\mathcal{P}} \|z_{\mathcal{P}}\|_2 = 0$. This establishes condition (9.11a). In this case (9.11b) is the formula

$$\lim B_{\mathcal{P}}(t;x,y) = B_\infty(t;x,y).$$

Since $1 + z_{\mathcal{P}}(k) = \langle x(t_{k-1}, t_k), y(t_{k-1}, t_k)\rangle$ we may conclude from Lemma 9.11 that

$$\lim_{\mathcal{P}} \prod_{I\in\mathcal{P}} \langle x_I, y_I\rangle = e^{B_\infty(t;x,y)}.$$

But for every partition $\mathcal{P}$ we have

$$\langle x, y\rangle = \prod_{I\in\mathcal{P}} \langle x_I, y_I\rangle\,,$$

and thus the preceding discussion implies that

$$\langle x, y\rangle = e^{B_\infty(t;x,y)}$$

which is the required formula (9.10). □

Remark 9.13. The argument we have given yields a somewhat stronger result involving *sequential* convergence. For each $t > 0$, let $\mathcal{P}_{1,t} \leq \mathcal{P}_{2,t} \leq \dots$ be any increasing sequence of partitions of the interval $[0, t]$ with the property

$$\lim_{n\to\infty} |\mathcal{P}_{n,t}| = 0,$$

for every t. Then for every $x, y \in D(t)$ satisfying $\langle x, e_t\rangle = \langle y, e_t\rangle = 1$, we claim that

$$L^e(t; x, y) = \lim_{n\to\infty} \sum_{I\in\mathcal{P}_{n,t}} (\langle x_I, y_I\rangle - 1). \tag{9.14}$$

To see that, fix x, y and define $B'_n(t; x, y)$ by

$$B'_n(t; x, y) = \sum_{I\in\mathcal{P}_{n,t}} (\langle x_I, y_I\rangle - 1).$$

(9.5) and (9.7) imply that the limit

$$B'_\infty(t; x, y) = \lim_{n\to\infty} B'_n(t; x, y)$$

exists. Moreover, we may apply Lemma 9.11 as in the preceding argument to conclude that

$$\langle x, y\rangle = e^{B'_\infty(t;x,y)}.$$

Finally, the estimates of Proposition 9.8 are valid for $B'_\infty(t; x, y)$ as well as for $B_\infty(t; x, y)$. Thus for any pair of left-decomposable sections u, v satisfying $\langle u_s, e_s\rangle = \langle v_s, e_s\rangle = 1$, the function

$$t \in (0, \infty) \mapsto B'_\infty(t; u_t, v_t)$$

is continuous and tends to 0 as $t \to 0+$. It follows that

$$B'_\infty(t; u_t, v_t) = B_\infty(t; u_t, v_t) = L^e(t; u_t, v_t)$$

for all t. Fixing t and choosing u, v appropriately, we obtain (9.14).

We will make use of formula (9.14) in the sequel.

10. Existence of measurable propagators.

The reader may have noticed that none of the results of sections 5 through 9 made any reference to measurability or to measurable sections, even though there is a natural Borel structure on the product system E. Measurability was simply not an issue in those matters. On the other hand, in the applications that will be discussed in part III it will be necessary do deal with measurable elements of D, and with a measurable reference section e; this is necessary in order to satisfy the measurability hypothesis of Definition 2.2. The existence of sufficiently many measurable elements of D is established in the following result.

Theorem 10.1. *Let $t_0 > 0$ and let $u \in E(t_0)$ be a nonzero decomposable vector. Then there is a left-coherent decomposable Borel section $t \in (0, \infty) \to x_t \in D(t)$ such that x_{t_0} is a scalar multiple of u.*

proof. We may assume without loss of generality that $\|u\| = 1$. Since u is decomposable, we can find elements a_s, $u_s \in E(s)$ such that

$$u = a_s u_{t_0 - s}, \qquad 0 < s < t_0; \tag{10.2}$$

and by renormalizing again if necessary we can arrange that $\|a_s\| = \|u_s\| = 1$. Set $a_{t_0} = u$. Then $\{a_s : 0 < s \le t_0\}$ is a left-decomposable family of unit vectors. The following result implies that we can choose a in a measurable way.

Proposition 10.3. *Let $t \in (0, t_0] \mapsto a_t \in E(t)$ be any left-coherent family of unit vectors. Then there is a function $t \in (0, t_0] \mapsto \lambda_t \in \mathbb{C}$ such that $|\lambda_t| = 1$ for every t and $t \mapsto \lambda_t a_t$ is a measurable section of $E \restriction_{(0, t_0]}$.*

Let us assume, for the moment, that the technical result 10.3 has been established. If we replace a_s with $\lambda_s a_s$ in (10.2) then we may assume that a_s is measurable in s. We can now define a measurable section $t \in (0, \infty) \mapsto x_t \in E(t)$ by

$$x_t = \begin{cases} \langle a_t, e_t \rangle^{-1} a_t, & \text{for } 0 < t \le t_0 \\ a_{t_0} e(t_0, t), & \text{for } t > t_0, \end{cases}$$

where $\{e(s, t) : 0 < s < t < \infty\}$ is the propagator associated with e. It is clear that $x \in D^e$ and that x_{t_0} is a scalar multiple of u, completing the proof of Theorem 10.1.

proof of 10.3. Consider the family of operators $\{P_s : 0 < s < t_0\} \subseteq \mathcal{B}(E(t_0))$ defined by

$$P_s(\xi) = a_s \cdot L_{a_s}^*(\xi), \qquad \xi \in E(t_0)$$

(see the proof of Theorem 2.1 for a discussion of operators of the form $L_z^*, z \in E$). Notice that since $E(t_0)$ is spanned by $E(s)E(t_0 - s)$, P_s is uniquely determined by its action on decomposable vectors $\xi_1 \xi_2$, $\xi_1 \in E(s)$, $\xi_2 \in E(t_0 - s)$:

$$P_s(\xi_1 \xi_2) = \langle \xi_1, a_s \rangle \, a_s \xi_2.$$

This also shows that if we identify $E(t_0)$ with $E(s) \otimes E(t_0 - s)$ then P_s becomes

$$[a_s] \otimes \mathbf{1}$$

$[a_s]$ denoting the projection onto the one-dimensional subspace $\mathbb{C} \cdot a_s$ of $E(s)$ and $\mathbf{1}_r$ denoting the identity operator in $\mathcal{B}(E(r))$. Now since $\{a_s : 0 < s < t_0\}$ is left-coherent, the family of projections $\{P_s : 0 < s < t_0\}$ satisfies

$$s_1 < s_2 \implies P_{s_2} \leq P_{s_1}$$

(see the proof of Theorem 2.1). Thus for any fixed $\xi \in E(t_0)$,

$$s \mapsto \langle P_s \xi, \xi \rangle$$

is a monotone decreasing function of $(0, t_0)$, and hence measurable. By polarization it follows that $\langle P_s \xi, \eta \rangle$ is measurable in s for any $\xi, \eta \in E(t_0)$. Hence

$$s \in (0, t_0) \mapsto P_s \in \mathcal{B}(E(t_0))$$

is a measurable projection-valued operator function.

For each $0 < s < t_0$, let Q_s denote the rank-on projection $Q_s = [a_s] \in \mathcal{B}(E(s))$. We have just seen that

$$s \mapsto Q_s \otimes \mathbf{1}_{t_0 - s} \tag{10.5}$$

is measurable and we claim now that Q itself is measurable. Equivalently, we claim that for any pair of measurable sections $\xi_s, \eta_s \in E(s)$, the complex valued function $s \in (0, t_0) \mapsto \langle Q_s \xi_s, \eta_s \rangle$ is measurable. To see that we choose any measurable section $t \mapsto u_t \in E(t)$ of unit vectors and write

$$\langle Q_s \xi_s, \eta_s \rangle = \langle Q_s \xi_s, \eta_s \rangle \langle u_{t_0-s}, u_{t_0-s} \rangle = \langle Q_s \otimes \mathbf{1}_{t_0-s}(\xi_s u_{t_0-s}), \eta_s u_{t_0-s} \rangle,$$

for $0 < s < t_0$. The right side is measurable in s by (10.6) and the fact that both $\xi_s u_{t_0-s}$ and $\eta_s u_{t_0-s}$ are measurable functions of s.

Finally, since $s \in (0, t_0) \mapsto Q_s \in \mathcal{B}(E(s))$ is a measurable family we claim that there is a measurable section $s \in (0, t_0) \mapsto b_s \in E(s)$ satisfying $\|b_s\| = 1$ for every s and

$$Q_s b_s = b_s, \qquad 0 < s < t_0. \tag{10.6}$$

To prove (10.6) we choose a measurable basis for the family of Hilbert spaces $p : E \to (0, \infty)$. That is, we find a sequence of measurable sections $e_n : t \in (0, \infty) \mapsto e_n(t) \in E(t)$, $n = 1, 2, \dots$ such that $\{e_1(t), e_2(t), \dots\}$ is an orthonormal basis for $E(t)$ for every $t > 0$. This is possible because of the last axiom for product systems [2, (1.8) *iii*]. For every $t > 0$ we must have $Q_t e_n(t) \neq 0$ for some n, and we define $n(t)$ to be the smallest such n. Now for every positive integer k we have

$$\{t \in (0, \infty) : n(t) > k\} = \bigcap_{i=1}^{k} \{t \in (0, \infty) : \langle Q_t e_i(t), e_i(t) \rangle = 0\},$$

and the right side is a Borel set since each function $t \mapsto \langle Q_t e_i(t), e_i(t) \rangle$ is measurable. It follows that the function $t \in (0, \infty) \mapsto n(t) \in \mathbb{R}$ is measurable. Hence $\xi_t = e_{n(t)}$ defines a measurable section of E having the property that $Q_t \xi_t \neq 0$ for every $t > 0$. Thus we obtain a section b as required by (10.6) by setting

$$b_t = \|Q_t \xi_t\|^{-1} Q_t \xi_t, \qquad 0 < t < t_0.$$

Now since Q_t is the projection onto the one-dimensional space $\mathbb{C} a_t$ it follows that there is a complex number λ_t such that $b_t = \lambda_t a_t$. Since $\|a_t\| = \|b_t\| = 1$ we have $|\lambda_t| = 1$, completing the proof of Proposition 10.3. $\square$

Part III. Applications

In the following two sections we apply the preceding results to classify certain product systems and certain E_0-semigroups.

11. Decomposable continuous tensor products. A product system $p : E \to (0, \infty)$ is said to be *decomposable* if for every $t > 0$, $E(t)$ is the closed linear span of the set $D(t)$ of all its decomposable vectors. It is easy to see that if this condition is satisfied for a single $t_0 > 0$ then it is satisfied for every $t > 0$.

Theorem 11.1. *A decomposable product system is either isomorphic to the trivial product system with one-dimensional spaces $E(t)$, $t > 0$, or it is isomorphic to one of the standard product systems E_n, $n = 1, 2, \dots, \infty$.*

Remarks. We recall that any product system with one-dimensional fibers $E(t)$ for every $t > 0$ is isomorphic to the trivial product system $p : Z \to (0, \infty)$, where $Z = (0, \infty) \times \mathbb{C}$ with multiplication $(s, z)(t, w) = (s + t, zw)$, with the usual inner product on $\mathbb{C}$, and with projection $p(t, z) = t$ [6, Corollary of Prop. 2.3] and [7].

proof of Theorem 11.1.

By Theorem 10.3, we can find a Borel section $t \in (0, \infty) \mapsto e(t) \in D(t)$. By replacing e_t with $e_t/\|e_t\|$ if necessary we may assume that $\|e_t\| = 1$ for every t. Let $L^e : \Delta^2 \to \mathbb{C}$ be the function provided by Theorem 8.2 which satisfies

$$e^{L^e(x,y)} = \frac{\langle x, y\rangle}{\langle x, e_t\rangle \langle e_t, y\rangle} \tag{11.2}$$

for every $x, y \in D(t)$, $t > 0$. We will show first that (Δ, L^e) is a metric path space; that is, L^e is an additive form on Δ^2.

Indeed, most of that assertion follows immediately from the results of Parts I and II. Theorem 9.2 implies that L^e restricts to a positive definite function on $\Delta(t) \times \Delta(t)$ for every $t > 0$, and because of Propositions 9.5 and 9.7 taken together with with formula 9.14, L^e must satisfy the separability condition of Definition 2.6. Proposition 8.8 shows that L^e is additive with defect function $\psi : \Delta \times \Delta \to \mathbb{C}$ of the form $\psi(x, y) = \psi_s(t; y)$ for every $x \in D(s)$, $y \in D(t)$, $s, t > 0$. Thus we need only establish the measurability criterion of Definition 2.2.

Notice that Definition 2.2 makes reference to the propagator $\{\dot z(r, s) : 0 \le r < s \le t\}$ associated with an element $\dot z \in \Delta(t)$. While this propagator in the path space Δ is uniquely determined by $\dot z$, an element $z \in D(t)$ does not determine a unique propagator $\{z(r, s) \in D(s - r) : 0 \le r < s \le t\}$ because of the failure of unique factorization in $\{D(t) : t > 0\}$. Nevertheless, any propagator in Δ can be lifted to a *measurable* propagator in $\{D(t) : t > 0\}$. More precisely, given any element $z \in D(t)$ then Theorem 10.1 provides a left-coherent *measurable* family $\{z_s \in D(s) : 0 < s \le t\}$ for which z_t is a scalar multiple of z. Recalling that a left-coherent family gives rise to a unique propagator because of the left cancellation law in $\{D(t) : t > 0\}$, we conclude that the propagator $\{z(r, s) \in D(s - r) : 0 \le r < s \le t\}$ associated with $\{z_s : 0 < s \le t\}$ projects to the required propagator in Δ, viz

$$\dot z_s = \dot z_r \dot z(r, s), \qquad 0 < r < s \le t,$$

and of course $\dot z(0, s) = \dot z_s$ for $0 < s \le t$.

In order to establish the measurability criterion of Definition 2.2, choose T_1, T_2 satisfying $0 < T_1 < T_2$ and choose $x \in D(T_1)$, $y \in D(T_2)$. By the preceding remarks, we may find a measurable left-coherent section $y_s \in D(s)$, $0 < s \leq T_2$ whose propagator projects to the propagator of $\dot{y} \in \Delta(T_2)$. It suffices to show that the function

$$\lambda \in (0, T_2 - T_1) \mapsto L^e(T_1; x, y(\lambda, \lambda + T_1)) \tag{11.3}$$

is a complex-valued Borel function.

To prove (11.3) we will make use of the fact that L^e is the limit of a *sequence* of functions for which the fact of measurability is obvious. Let $\mathcal{P}_1, \mathcal{P}_2, \dots$ be a sequence of finite partitions of the interval $[0, T_1]$ such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ and such that the norms $|\mathcal{P}_n|$ tend to zero as $n \to \infty$. By (9.14) we may conclude that for every $u, v \in D(T_1)$ satisfying $\langle u, e_{T_1} \rangle = \langle v, e_{T_1} \rangle = 1$, we have

$$L^e(T_1; u, v) = \lim_{n \to \infty} \sum_{I \in \mathcal{P}_n} (\langle u_I, v_I \rangle - 1).$$

Taking

$$\begin{aligned} u &= \langle x, e_{T_1} \rangle^{-1} x \qquad \text{and} \\ v &= v_\lambda = \langle y(\lambda, \lambda + T_1), e_{T_1} \rangle^{-1} y(\lambda, \lambda + T_1), \end{aligned}$$

we see that the left side of (11.3) is exhibited as the limit of a convergent sequence of functions

$$f_n(\lambda) = \sum_{I \in \mathcal{P}_n} (\langle u_I, (v_\lambda)_I \rangle - 1).$$

Thus it suffices to show that each f_n is Borel-measurable on $0 < \lambda < T_2 - T_1$. In order to see that, choose an interval $I = (a, b] \subseteq (0, T_1]$ for which $a < b$ and look at the inner product $\langle u_I, (v_\lambda)_I \rangle$. Noting that

$$\begin{aligned} y(\lambda, \lambda + T_1) &= y(\lambda, \lambda + a) y(\lambda + a, \lambda + b) y(\lambda + b, \lambda + T_1) \\ e_{T_1} &= e_a e(a, b) e(b, T_1), \end{aligned}$$

we can write down an obvious propagator for v_λ and we find that

$$v_\lambda(a, b) = \langle y(\lambda + a, \lambda + b), e(a, b) \rangle^{-1} y(\lambda + a, \lambda + b).$$

It follows that $f_n(\lambda)$ is a finite linear combination of functions of the form

$$\lambda \in (0, T_2 - T_1) \mapsto \frac{\langle u(a, b), y(\lambda + a, \lambda + b) \rangle}{\langle e(a, b), y(\lambda + a, \lambda + b)) \rangle} - 1. \tag{11.4}$$

Now for any element $w \in D(b - a)$ we have

$$\begin{aligned} \langle y_{\lambda + a} w, y_{\lambda + b} \rangle &= \langle y_{\lambda + a} w, y_{\lambda + a} y(\lambda + a, \lambda + b) \rangle \\ &= \|y_{\lambda + a}\|^2 \langle w, y(\lambda + a, \lambda + b) \rangle . \end{aligned}$$

Thus the right side of (11.4) can be written

$$\frac{\langle y_{\lambda + a} u(a, b), y_{\lambda + b} \rangle}{\langle y_{\lambda + a} e(a, b), y_{\lambda + b} \rangle} - 1.$$

This is obviously a measurable function of λ because $s \mapsto y_s$ is a measurable section, left multiplication by a fixed element of E is a measurable mapping of E into itself, and because the inner product $\langle \cdot, \cdot \rangle : E^2 \to \mathbb{C}$ is measurable

Thus we have established the fact that (Δ, L^e) is a metric path space. By Theorem 4.3 there is a separable Hilbert space $\mathcal{C}$, a function $\rho : \Delta \to \mathbb{C}$, and a logarithm mapping

$$\log : \Delta \to \mathcal{P}_{\mathcal{C}}$$

such that $\log(xy) = \log(x) \boxplus \log(y)$ for every $x, y \in \Delta$ and

$$L^e(t; x_1, x_2) = \langle \log(x_1), \log(x_2) \rangle + \rho(x_1) + \overline{\rho(x_2)} \tag{11.5}$$

for all $x_1, x_2 \in \Delta(t)$, $t > 0$. In fact, in order to obtain the necessary measurability properties, we must use the specific function ρ defined in the proof of Theorem 4.3.

Now when the path space $\mathcal{P}_{\mathcal{C}}$ is exponentiated, it gives rise to the standard product system $E_{\mathcal{C}}$. In more detail, consider the symmetric Fock space $H_{\mathcal{C}}$ over the one-particle space $L^2((0, \infty); \mathcal{C})$, and consider the exponential map

$$\exp : L^2((0, \infty); \mathcal{C}) \to H_{\mathcal{C}}$$

defined by

$$\exp(f) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

For every $t > 0$ we define

$$E_{\mathcal{C}}(t) = \overline{\operatorname{span}}\{\exp(f) : f \in \mathcal{P}_{\mathcal{C}}(t)\}.$$

$E_{\mathcal{C}}$ is the total space of this family of Hilbert spaces, with multiplication $\xi \in E_{\mathcal{C}}(s), \eta \in E_{\mathcal{C}}(t) \mapsto \xi\eta \in E_{\mathcal{C}}(s+t)$ defined uniquely by requiring that the generating vectors should multiply thus:

$$\exp(f)\exp(g) = \exp(f \boxplus g), \qquad f \in \mathcal{P}_{\mathcal{C}}(s), g \in \mathcal{P}_{\mathcal{C}}(t).$$

We will use (11.5) to construct an isomorphism of product systems $W : E \to E_{\mathcal{C}}$. But in order to define W it is necessary to solve another cohomological problem. The result is summarized as follows. We write

$$p : D = \{(t, x) : x \in D(t), t > 0\} \to (0, \infty)$$

for the fiber space determined by the family of sets $D(t), t > 0$ with projection $p(t, x) = t$. Notice that D carries a natural Borel structure as a subspace of E.

Theorem 11.6. *There is a Borel-measurable function $u : (0, \infty) \to \mathbb{C}$ satisfying $|u(t)| = 1$ for every $t > 0$, such that the function $f : D \to \mathbb{C}$ defined by*

$$f(x) = u(t)\, \langle x, e_t \rangle\, e^{\rho(x)}, \qquad x \in D(t), \quad t > 0$$

is multiplicative: $f(xy) = f(x)f(y)$, $x \in D(s), y \in D(t)$.

proof of Theorem 11.6. Let $f_0(x) = \langle x, e_t \rangle\, e^{\rho(x)}$. Notice first that f_0 is Borel-measurable. Indeed, recalling the formula for $\rho(x)$ given in the proof of Theorem 4.3, we have for every $x \in D(s)$ and $s > 0$,

$$\rho(x) = \langle [x] - [e_s], \phi_s \rangle + L^e(s; x, e_s) - \frac{1}{2}(L^e(s; e_s, e_s) + \|\phi_s\|^2), \tag{11.7}$$

Because of the representation of L^e as a sequential limit in (9.14) we see that $L^e : D^2 \to \mathbb{C}$ is a Borel function, and it follows that $x \in D(s) \mapsto [x] - [e_s]$ defines a Borel map of D into the Hilbert space H_∞. Finally, since $s \in (0, \infty) \mapsto \phi_s$ is a Borel function taking values in the space of all locally square integrable $\mathcal{C}$-valued functions on $(0, \infty)$, we see that the right side of (11.7) defines a Borel function on the subspace D of E. Since the inner product is a Borel function on E, it follows that

$$f_0(x) = \langle x, e_s \rangle \, e^{\rho(x)}, \qquad x \in D(s)$$

defines a complex-valued Borel function on D.

We consider the associated "coboundary" $c : D \times D \to \mathbb{C}$:

$$c(x, y) = \frac{f_0(xy)}{f_0(x) f_0(y)}.$$

Note that c is a Borel function as well, since the multiplication operation of E is Borel measurable. We will show that for every $s, t > 0$, $x_i \in D(s)$, $y_i \in D(t)$,

$$|c(x_1, y_1)| = 1, \tag{11.8.1}$$

$$c(x_1, y_1) = c(x_2, y_2). \tag{11.8.2}$$

Assume for the moment that the equations (11.8) have been established. It follows that there is a function

$$c_0 : (0, \infty) \times (0, \infty) \to \{z \in \mathbb{C} : |z| = 1\}$$

such that for all $x \in D(s)$, $y \in D(t)$ we have

$$c(x, y) = c_0(s, t).$$

c_0 is clearly measurable because $c_0(s, t) = c(e_s, e_t)$, and $r \mapsto e_r \in D(r)$ is a measurable section. We will then show that c_0 satisfies the multiplier equation,

$$c_0(r, s + t) c_0(s, t) = c_0(r + s, t) c_0(r, s), \tag{11.9}$$

for every $r, s, t > 0$. By [6, Corollary of Propostion 2.3], there is a Borel function $u : (0, \infty) \to \{z \in \mathbb{C} : |z| = 1\}$ such that

$$c_0(s, t) = \frac{u(s) u(t)}{u(s + t)}, \qquad s, t > 0.$$

Once we have u it is clear that the function $f(x) = u(t) f_0(x)$, $x \in D(t)$, $t > 0$ satisfies

$$f(xy) = f(x) f(y).$$

Thus, we must prove the formulas (11.8) and (11.9). For $t > 0$, consider the function $W_t : D(t) \to E_{\mathcal{C}}(t)$ defined by

$$W_t(x) = f_0(x) \exp \log(x) = \langle x, e_t \rangle \, e^{\rho(x)} \exp \log(x)$$

We claim that for $x_1, x_2 \in D(t)$ we have

$$\langle W_t(x_1), W_t(x_2)\rangle = \langle x_1, x_2\rangle . \tag{11.10}$$

Indeed, using Theorem 4.3 we have

$$\begin{aligned}\langle \exp\log(x_1), \exp\log(x_2)\rangle =& e^{\langle \log(x_1), \log(x_2)\rangle} = \exp(L^e(t; x_1, x_2) - \rho(x_1) - \overline{\rho(x_2)}) \\ =& \frac{\langle x_1, x_2\rangle}{\langle x_1, e_t\rangle \langle e_t, x_2\rangle} e^{-(\rho(x_1) + \overline{\rho(x_2)})}.\end{aligned}$$

Thus the left side of (11.10) is

$$f_0(x_1)\overline{f_0(x_2)}\,\langle \exp\log(x_1), \exp\log(x_2)\rangle = \langle x_1, e_t\rangle\,\overline{\langle x_2, e_t\rangle}\frac{\langle x_1, x_2\rangle}{\langle x_1, e_t\rangle \langle e_t, x_2\rangle} = \langle x_1, x_2\rangle ,$$

as asserted.

The family $\{W_t : t > 0\}$ obeys the following multiplicative rule. For $x \in D(s)$, $y \in D(t)$, we claim

$$W_{s+t}(xy) = c(x, y)W_s(x)W_t(y). \tag{11.11}$$

Notice that the multiplication on the left side of (11.11) is performed in E and on the right side it is performed in $E_{\mathcal{C}}$. To check (11.11), recall that the multiplication in $E_{\mathcal{C}}$ is related to the operation $\boxplus$ by the following formula: if $f \in E_{\mathcal{C}}(s)$ and $g \in E_{\mathcal{C}}(t)$ then

$$\exp(x)\exp(g) = \exp(f \boxplus g).$$

Thus, using the additivity property of the logarithm mapping the left side of (11.11) can be rewritten

$$\begin{aligned}f_0(xy)\exp\log(xy) =& f_0(xy)\exp(\log(x) \boxplus \log(y)) = f_0(xy)\exp(\log(x))\exp(\log(y)) \\ =& \frac{f_0(xy)}{f_0(x)f_0(y)} W_s(x)W_t(y) = c(x, y)W_s(x)W_t(y).\end{aligned}$$

We claim next that for $x_i \in D(s)$, $y_i \in D(t)$ we have

$$c(x_1, y_1)\overline{c(x_2, y_2)} = 1. \tag{11.12}$$

To see this, choose $x_i \in D(s)$, $y_i \in D(t)$. Noting that by (11.10)

$$\begin{aligned}\langle x_1y_1, x_2y_2\rangle = \langle x_1, x_2\rangle \langle y_1, y_2\rangle =& \langle W_s(x_1), W_s(x_2)\rangle \langle W_t(y_1), W_t(y_2)\rangle \\ =& \langle W_s(x_1)W_t(y_1), W_s(x_2)W_t(y_2)\rangle ,\end{aligned}$$

we have

$$\begin{aligned}& c(x_1, y_1)\overline{c(x_2, y_2)}\,\langle x_1y_1, x_2y_2\rangle = \\ & c(x_1, y_1)\overline{c(x_2, y_2)}\,\langle W_s(x_1)W_t(y_1), W_s(x_2)W_t(y_2)\rangle = \\ & \langle c(x_1, y_1)W_s(x_1)W_t(y_1), c(x_2, y_2)W_s(x_2)W_t(y_2)\rangle = \\ & \langle W_{s+t}(x_1y_1), W_{s+t}(x_2y_2)\rangle = \langle x_1y_1, x_2y_2\rangle\end{aligned}$$

The claim follows after cancelling $\langle x_1y_1, x_2y_2\rangle \neq 0$.

Set $x_2 = x_1, y_2 = y_1$ in (11.12) to obtain $|c(x_1, y_1)| = 1$. Thus if we multiply through in (11.12) by $c(x_2, y_2)$ we obtain

$$c(x_1, y_1) = c(x_2, y_2),$$

hence (11.8.1) and (11.8.2) are established.

Now we can define a function $c_0 : (0,\infty) \times (0,\infty) \to \mathbb{C}$ by $c_0(s,t) = c(e_s, e_t)$. c_0 is a Borel function which, because of (11.11), obeys

$$W_{s+t}(xy) = c_0(s,t)W_s(x)W_t(y), \qquad x \in D(s), y \in D(t).$$

The latter formula implies that c_0 must satisfy the multiplier equation (11.9). Indeed, for $r, s, t > 0$ and $x \in D(r), y \in D(s), z \in D(t)$ we have

$$W_{r+s+t}(x(yz)) = c_0(r, s+t)W_r(x)W_{s+t}(yz) = c_0(r, s+t)c_0(s,t)W_r(x)W_s(y)W_t(z),$$

while

$$W_{r+s+t}((xy)z) = c_0(r+s, t)W_{r+s}(xy)W_t(z) = c_0(r+s, t)c_0(r, s)W_r(x)W_s(y)W_t(z).$$

Since $W_r(x)W_s(y)W_t(z) \neq 0$, equation (11.9) follows.

The argument given above can now be applied to complete the proof of Theorem 11.6 $\square$

We are now in position to write down an isomorphism $E \cong E_{\mathcal{C}}$. Choose u as in Theorem 11.6. For every $t > 0$, $x \in D(t)$ define $W_t(x) \in E_{\mathcal{C}}(t)$ by

$$W_t(x) = u(t)\,\langle x, e_t\rangle\, e^{\rho(x)} \exp(\log(x)).$$

Equation (11.10) implies that

$$\langle W_t(x_1), W_t(x_2)\rangle = \langle x_1, x_2\rangle\,.$$

Since $E(t)$ is spanned by $D(t)$, W_t can be extended uniquely to a linear isometry of $E(t)$ into $E_{\mathcal{C}}(t)$, and we will denote the extended mapping by the same letter W_t. The range of W_t is given by

$$W_t(E(t)) = \overline{\operatorname{span}}[\exp\log(D(t))] = E_{\mathcal{C}}(t),$$

because the set $\log(D(t))$ is a strongly spanning subset of $\mathcal{P}_{\mathcal{C}}(t)$. Thus the total map

$$W : E \to E_{\mathcal{C}}$$

is an isomorphism of families of Hilbert spaces.

Because of the way we chose the function u, the multiplication formula (11.11) simplifies to

$$W_{s+t}(xy) = W_s(x)W_t(y), \qquad x \in D(s), y \in D(t).$$

Using bilinearity and the fact that $D(r)$ spans $E(r)$ for every $r > 0$, the latter implies that W is a homomorphism of product structures in that

$$W_{s+t}(\xi\eta) = W_s(\xi)W_t(\eta), \qquad \xi \in E(s), y \in E(t).$$

In particular, W is a bijection of the standard Borel space E onto the standard Borel space $E_{\mathcal{C}}$. Thus to see that W is a Borel isomorphism (and therefore an isomorphism of product systems), it suffices to show that it is measurable. The proof of that is a routine variation on the argument presented in detail in [2, pp 55–57], and we omit it.

Finally, notice that it is possible that $\mathcal{C}$ is the trivial Hilbert space $\{0\}$. However, in this case $E_{\mathcal{C}}$ is the trivial product system with one-dimensional fibers. If $\mathcal{C} \neq \{0\}$ and we let n be the dimension of $\mathcal{C}$, then $n = 1, 2, \dots, \aleph_0$ and $E_{\mathcal{C}}$ is the standard product system E_n $\square$

12. Decomposable E_0-semigroups. Let $\alpha = \{\alpha_t; t \geq 0\}$ be an E_0-semigroup, and for every $t > 0$ let $\mathcal{E}(t)$ be the operator space

$$\mathcal{E}(t) = \{T \in \mathcal{B}(H) : \alpha_t(A)T = TA, \forall A \in \mathcal{B}(H)\}.$$

$\mathcal{E}(t)$ is a Hilbert space relative to the inner product defined on it by

$$T^*S = \langle S, T \rangle \mathbf{1}.$$

The family of Hilbert spaces

$$p : \mathcal{E} = \{(t, T) : t > 0, T \in \mathcal{E}(t)\} \to (0, \infty)$$

with projection $p(t, T) = t$ is actually a product system with respect to operator multiplication $(s, S)(t, T) = (s+t, ST)$ [2]. In particular, $\mathcal{E}(s+t)$ is the norm closed linear span of the set of all products $\{ST : S \in \mathcal{E}(s), T \in \mathcal{E}(t)\}$.

As with any product system, it makes sense to speak of decomposable elements of $\mathcal{E}(t)$; thus, an operator $T \in \mathcal{E}(t)$ is decomposable if, for every $0 < s < t$, T admits a factorization $T = AB$ where $A \in \mathcal{E}(s)$ and $B \in \mathcal{E}(t-s)$. The set of all decomposable operators in $\mathcal{E}(t)$ is denoted $\mathcal{D}(t)$. Let $H(t) = [\mathcal{D}(t)H]$ be the closed linear span of the ranges of all operators in $\mathcal{D}(t)$. The spaces $H(t)$ are obviously decreasing with t.

Definition 12.1. *$\alpha = \{\alpha_t : t \geq 0\}$ is called decomposable if $[\mathcal{D}(t)H] = H$ for some (and therefore every) $t > 0$.*

The following simple result shows that this terminology is consistent with the notion of decomposability for product systems.

Proposition 12.2. *An E_0-semigroup α is decomposable iff its associated product system is decomposable in the sense of section 11.*

proof. Suppose first that $\mathcal{E}$ is a decomposable product system. Then for every $t > 0$ $\mathcal{E}(t)$ is the norm-closed linear span of $\mathcal{D}(t)$. Since we have $[\mathcal{E}(t)H] = H$ for the product system of an arbitrary E_0-semigroup, it follows that $H = [\mathcal{D}(t)H]$ as well.

Conversely, assuming that α satisfies Definition 12.1 we pick $t > 0$ and an operator $T \in \mathcal{E}(t)$ such that T is orthogonal to $\mathcal{D}(t)$. Because of the definition of the inner product in $E(t)$ it follows that for every $S \in \mathcal{D}(t)$ we have

$$T^*S = \langle S, T \rangle \mathbf{1} = 0,$$

hence TH is orthogonal to $[\mathcal{D}(t)H] = H$, hence $T = 0$. Thus $\mathcal{D}(t)$ spans $\mathcal{E}(t)$ □

We can immediately deduce from Theorem 11.1 that the product system of a decomposable E_0-semigroup is either the trivial product system or it is isomorphic to one of the standard product systems E_n, $n = 1, 2, \dots, \infty$. Since the product system of an E_0-semigroup is a complete invariant for cocycle conjugacy and since the standard product systems E_n are associated with CCR flows (or CAR flows) [2], we can infer the following classification of E_0-semigroups as an immediate consequense of Theorem 11.1.

Theorem 12.3. *Let α be a decomposable E_0-semigroup acting on $\mathcal{B}(H)$ which is nontrivial in the sense that it cannot be extended to a group of automorphisms of $\mathcal{B}(H)$. Then α is cocycle conjugate to a CCR flow.*

Concluding remarks. Note that Theorem 12.3 implies that every decomposable E_0-semigroup α has plenty of intertwining semigroups...that is, semigroups of isometries $U = \{U_t : t \geq 0\}$ acting on H for which

$$\alpha_t(A)U_t = U_tA, \qquad A \in \mathcal{B}(H), t > 0.$$

Indeed, a decomposable E_0-semigroup must be completely spatial.

In a more philosophical vein, we conclude that *any construction of E_0-semigroups which starts with a path space cannot produce anything other than CCR flows and their cocycle perturbations.* For example, in [2 pp 14–16] we gave examples of product systems using Gaussian random processes and Poisson processes. The latter examples did not appear to contain enough units to be standard ones. However, a closer analysis showed that there were "hidden" units, and in fact there were enough of them so that these product systems were indeed standard. Theorems 11.1 and 12.4 serve to clarify this phenomenon because all examples constructed in this way from random processes are obviously decomposable.

We believe that Theorem 12.4 is analogous to the familiar description of representations of the compact operators (i.e., every representation is unitarily equivalent to a multiple of the identity representation), or to the Stone-von Neumann theorem. Certainly, the conclusion of 12.4 implies that decomposable E_0-semigroups exhibit "type I" behavior. What is interesting here is that the E_0-semigroups whose product systems are isomorphic to a given one E correspond bijectively with the "essential" representations of the spectral C^*-algebra $C^*(E)$ [3, section 3], [4], and [5]. This correspondence has the feature that unitary equivalence of representations of $C^*(E)$ corresponds to conjugacy of E_0-semigroups. More precisely, if π_1 and π_2 are two essential representations of $C^*(E)$ with corresponding E_0-semigroups α_1 and α_2, then α_1 and α_2 are conjugate iff there is a gauge automorphism γ of $C^*(E)$ such that π_2 is unitarily equivalent to $\pi_1 \circ \gamma$. The structure of the group of gauge automorphisms has been explicitly calculated for the standard examples $C^*(E_n)$, $n = 1, 2, \dots, \infty$ [2, section 8].

In the case of decomposable E_0-semigroups the spectral C^*-algebra is a standard one $C^*(E_n)$, $n = 1, 2, \dots, \infty$. These examples are continuous analogues of the Cuntz algebra $\mathcal{O}_\infty$ [9], and are far from being type I C^*-algebras. Nevertheless, if we agree to identify two representations of $C^*(E_n)$ up to unitary equivalence modulo cocycle perturbations (that is to say, up to cocycle perturbations of the associated E_0-semigroups), as well as up to the internal action of the gauge group, then the resulting set of equivalence classes of representations is smooth: it is parameterized by a single integer $n = 1, 2, \dots, \infty$. The integer n is, of course, the numerical index of the associated E_0-semigroup.

On the other hand, we remind the reader that Powers has constructed examples of E_0-semigroups that are of type II (they have some intertwining semigroups but not enough of them) [27], and others that are of type III (they have no intertwining semigroups whatsoever) [25]. None of these more exotic E_0-semigroups can be decomposable. The structure of the product systems associated with such E_0-semigroups remains quite mysterious.

References

1. Araki, H. and Woods, E. J., *Complete Boolean algebras of type I factors*, Publ. RIMS (Kyoto University) **2, ser. A, no. 2** (1966), 157–242.
2. Arveson, W., *Continuous analogues of Fock space*, Memoirs Amer. Math. Soc. **80 no. 3** (1989).
3. ______, *Continuous analogues of Fock space II: the spectral C^*-algebra*, J. Funct. Anal. **90** (1990), 138–205.
4. ______, *Continuous analogues of Fock space III: singular states*, J. Oper. Th. **22** (1989), 165–205.
5. ______, *Continuous analogues of Fock space IV: essential states*, Acta Math. **164** (1990), 265–300.
6. ______, *An addition formula for the index of semigroups of endormorphisms of $\mathcal{B}(H)$*, Pac. J. Math. **137** (1989), 19–36.
7. ______, *Quantizing the Fredholm index*, Operator Theory: Proceedings of the 1988 GPOTS-Wabash conference (Conway, J. B. and Morrel, B. B., ed.), Pitman research notes in mathematics series, Longman, 1990.
8. ______, *The spectral C^*-algebra of an E_0-semigoup*, Operator Theory Operator Algebras and applications, Proc. Symp. Pure Math. (Arveson, W. and Douglas, R. G., ed.), vol. 51, part I, 1990, pp. 1–15.
9. ______, *C^*-algebras associated with sets of semigroups of isometries*, Int. J. Math. **2, no. 3** (1991), 235–255.
10. Arveson, W. and Kishimoto, A., *A note on extensions of semigroups of $*$-endomorphisms*, Proc. A. M. S. **116, no 3** (1992), 769–774.
11. Arveson, W., *E_0-semigroups and quantum field theory*, to appear.
12. Banach, S., *Théorie des Opérations Linéaires*, Chelsea, 1978.
13. Bratteli, O. and Robinson, D. W., *Operator algebras and quantum statistical mechanics I, II*, Springer-Verlag, 1989.
14. Connes, A., *Une classification des facteurs de type III*, Ann. Scient. Ecole Norm. Sup. **6 , fasc. 2 ser. 4e** (1973), 133–253.
15. ______, *Non Commutative Geometry*, Academic Press (to appear).
16. Cuntz, J., *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.
17. ______, *K-theory for certain C^*-algebras*, Ann. Math. **113** (1981), 181–197.
18. Guichardet, A., *Symmetric Hilbert spaces and related topics*, Springer-Verlag Lecture Notes in Mathematics 261, 1972.
19. Kuratowski, K., *Topologie I*, Warszawa-Lwów, 1933.
20. Mackey, G. W., *Unitary representations of group extensions I*, Acta Math. **99** (1958), 265–311.
21. Murray, F. J. and von Neumann, J., *On rings of operators*, Ann. Math. **37** (1936), 116–229.
22. von Neumann, J., *On infinite direct products*, Compositio Math. **6** (1938), 1–77.
23. Pedersen, G. K., *C^*-algebras and their automorphism groups*, Academic Press, 1979.
24. Powers, R. T., *An index theory for semigroups of endomorphisms of $\mathcal{B}(H)$ and type II factors*, Can. J. Math. **40** (1988), 86–114.
25. ______ A non-spatial continuous semigroup os $*$-endomorphisms of $\mathcal{B}(H)$, Publ. RIMS (Kyoto University) **23** (1987), 1053–1069.
26. ______ On the structure of continuous spatial semigroups of $*$-endomorphisms of $\mathcal{B}(H)$, Int. J. Math. **2, no 3** (1991), 323–360.
27. ______, *New examples of continuous spatial semigroups of endomorphisms of $\mathcal{B}(H)$*, (preprint 1994).
28. Powers, R. T. and Robinson, D., *An index for continuous semigroups of $*$-endomorphisms of $\mathcal{B}(H)$*, J. Funct. Anal. **84** (1989), 85–96.
29. Powers, R. T. and Price, G, *Continuous spatial semigroups of $*$-endomorphisms of $\mathcal{B}(H)$*, Trans. A. M. S. (to appear).
30. Reed, Michael and Simon, Barry, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness*, Academic Press, 1975.