MINIMAL $E_0$-SEMIGROUPS

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Abstract. It is known that every semigroup of normal completely positive maps of a von Neumann can be “dilated” in a particular way to an $E_0$-semigroup acting on a larger von Neumann algebra. The $E_0$-semigroup is not uniquely determined by the completely positive semigroup; however, it is unique (up to conjugacy) provided that certain conditions of minimality are met. Minimality is a subtle property, and it is often not obvious if it is satisfied for specific examples even in the simplest case where the von Neumann algebra is $B(H)$.

In this paper we clarify these issues by giving a new characterization of minimality in terms of projective cocycles and their limits. Our results are valid for semigroups of endomorphisms acting on arbitrary von Neumann algebras with separable predual.
1. Dilations and compressions. Let $M$ be a von Neumann algebra with separable predual. When it is convenient to do so, we will consider that $M$ is a von Neumann subalgebra of the algebra $\mathcal{B}(H)$ of all bounded operators on a separable Hilbert space $H$ which contains the identity operator $1$. The separability of $H$ will be essential for some of the results below.

By an $E_0$-semigroup acting on $M$ we mean a family of normal $*$-endomorphisms $\alpha = \{\alpha_t : t \geq 0\}$ of $M$ satisfying $\alpha_t(1) = 1$ for every $t \geq 0$, which obeys the semigroup property $\alpha_{s+t} = \alpha_s \alpha_t$, and which is continuous in the sense that for every $a \in M$ and every pair of vectors $\xi, \eta \in H$, the function $t \in [0, \infty) \mapsto \langle \alpha_t(a)\xi, \eta \rangle$ is continuous. We will also consider semigroups $\phi = \{\phi_t : t \geq 0\}$ of normal completely positive maps acting on certain von Neumann subalgebras $N \subseteq M$. These subalgebras will normally not contain the unit of $M$; but we will require that such semigroups $\phi$ be unital in the sense that $\phi_t(1_N) = 1_N$, $t \geq 0$, and that they should satisfy the natural continuity property cited above. We will refer to such a semigroup $\phi_t : N \to N$ simply as a completely positive semigroup.

$\alpha$ can be compressed to certain hereditary subalgebras of $M$ so as to give a completely positive semigroup as follows. Let $M_0 = pM_0$ be a hereditary von Neumann subalgebra of $M$ with unit $p$. The natural projection

$$E_0 : M \to M_0$$

of $M$ onto $M_0$ is defined by $E_0(a) = pap$. $E_0$ carries the unit of $M$ to that of $M_0$, and we have

$$E_0(axb) = aE_0(x)b, \quad a, b \in M_0, x \in M.$$ Fix $t \geq 0$. It is an elementary exercise to show that in order for there to exist a linear map $\phi_t : M_0 \to M_0$ satisfying $E_0 \circ \alpha_t = \phi_t \circ E_0$ it is necessary and sufficient that $\alpha_t(p) \geq p$. Thus we will be concerned with hereditary subalgebras $M_0 = pM_0$ for which the projection $p$ is increasing in the sense that

$$\alpha_t(p) \geq p, \quad t \geq 0.$$ (1.1)

In this case one can define a family $\phi = \{\phi_t : t \geq 0\}$ of completely positive maps of $M_0$ by compressing each map $\alpha_t$ to $M_0$,

$$\phi_t(a) = p\alpha_t(a)p, \quad a \in M_0, t \geq 0.$$ (1.2)

Since $\phi_t(p) = p$, we may consider $\phi_t$ to be a unital map of $M_0$. Note too that the family $\phi$ has the semigroup property $\phi_{s+t} = \phi_s \phi_t$ for $s, t \geq 0$. Indeed, since $p\alpha_s(p) = \alpha_s(p)p = p$ by (1.1) we have

$$\phi_s \phi_t(a) = p\alpha_s(p)\alpha_t(a)p = p\alpha_s(p)\alpha_s(a)\alpha_t(p) = \phi_{s+t}(a),$$

for every $a \in M_0$. Thus $\phi$ is a completely positive semigroup acting on $M_0$. Throughout this paper we will be concerned with properties of completely positive semigroups which can be obtained from a fixed $E_0$-semigroup in this particular way.
Definition 1.3. Let $p$ be a projection in $M$ satisfying (1.1), and let $pMp$ be the corresponding hereditary subalgebra. The completely positive semigroup $\phi = \{\phi_t : t \geq 0\}$ defined on $pMp$ by (1.2) is called a compression of $\alpha$, and $\alpha$ is called a dilation of $\phi$.

We emphasize that the notion of a compression of $\alpha$ to a subalgebra has meaning only when the subalgebra is (1) a hereditary subalgebra $pMp$ of $M$ and (2) $p$ is a projection satisfying (1.1). It is possible to conjure up other definitions of “compressions” of $\alpha$ and “dilations” of $\phi$. For example, one might imagine using a conditional expectation from $M$ onto a unital subalgebra $N$ of $M$ to attempt to define a semigroup of completely positive maps of $N$ under appropriate conditions. While there has been some limited success for such endeavors [8],[9], a recent theorem of B. V. R. Bhat [5] has led us to the conclusion that the proper context for this kind of dilation theory is the context of Definition 1.3.

More precisely, Bhat has shown that for every completely positive semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on a von Neumann algebra $M_0$, there is a larger von Neumann algebra $M$ containing $M_0$ as a hereditary subalgebra $M_0 = pMp$ and an $E_0$-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $M$ which satisfies $\alpha_t(p) \geq p$ for every $t$ and is such that $\phi$ is obtained from $\alpha$ by compression. Thus we have a dilation theory which resembles the more familiar dilation theory for operator semigroups which asserts that every semigroup of contraction operators on a Hilbert space can be dilated to a semigroup of isometries on a larger Hilbert space.

In the case of operator semigroups there is a simple notion of minimal isometric dilation, and two minimal isometric dilations of the same contraction semigroup are naturally unitarily equivalent. There is an analogous notion of minimality in the current setting and there is an analogous uniqueness result for the minimal $E_0$-semigroup dilation of a completely positive semigroup [5], [6]. However, these considerations for completely positive semigroups and their $E_0$-semigroup dilations are much more subtle than their counterparts in operator theory. To illustrate the level of subtlety, recall that a semigroup of contraction operators can actually be dilated further to a semigroup of unitary operators. That is because any semigroup of isometries acting on a Hilbert space is the restriction to an invariant subspace of a semigroup of unitary operators acting on a larger Hilbert space. The unitary semigroup is unique (up to a natural unitary equivalence) if the invariant subspace is “minimal”. Nothing like that is true in this setting. Indeed, an $E_0$-semigroup acting on a type I factor $M$ does not have a natural extension to a semigroup of automorphisms of a larger type I factor which contains $M$ as a unital subfactor. It is true that every $E_0$-semigroup can be so extended, but the construction of the extension is quite indirect and there is apparently no uniqueness of such extensions (see [2], or see [4] for a different proof). Finally, the notion of minimality introduced in [5], [6] is geared to quantum probability theory and does not lend itself readily to the natural questions that arise in the theory of $E_0$-semigroups.

The purpose of this paper is to clarify the issue of minimality for $E_0$-semigroups acting on arbitrary von Neumann algebras, and to give a new characterization of minimality in terms of the natural objects of operator algebras.

Let $p$ be an increasing projection in $M$ and let $\phi = \{\phi_t : t \geq 0\}$ be the compression of $\alpha$ to $pMp$. Notice that $\phi$ is itself an $E_0$-semigroup (acting on $pMp$) iff for every $t \geq 0$ we have

\begin{equation}
\phi_t(ab) = \phi_t(a)\phi_t(b), \quad a, b \in pMp.
\end{equation}
Definition 1.5. Let $\alpha$ be an $E_0$-semigroup acting on a von Neumann algebra $M$. A compression $\phi$ of $\alpha$ to a hereditary subalgebra which satisfies property (1.4) is called multiplicative.

Suppose that $\phi$ is a compression of an $E_0$-semigroup $\alpha$ to a hereditary subalgebra $pMp$ of $M$, and that $q$ is an increasing projection such that $q \geq p$ and the compression of $\alpha$ to $qMq$ is multiplicative. Then may consider that the compression of $\alpha$ to the intermediate subalgebra $qMq$ is itself an $E_0$-semigroup which has $\phi$ as a compression.

Definition 1.6. Let $\alpha$ be an $E_0$-semigroup acting on a von Neumann algebra $M$, let $p$ be an increasing projection in $M$, and let $\phi$ be the completely positive semigroup on $pMp$ obtained by compression. $\alpha$ is said to be minimal over $\phi$ if the only increasing projection $q \in M$ which satisfies $q \geq p$, and is such that the compression of $\alpha$ to $qMq$ is multiplicative, is the projection $q = 1$.

In order to discuss minimality further, one needs to know more about increasing projections which define multiplicative compressions. Here is the simplest class of examples. Let $p$ be a projection of $M$ which is fixed under $\alpha$ in the sense that

$$\alpha_t(p) = p, \quad t \geq 0.$$ 

In this case it is clear that the compression of $\alpha$ to the hereditary subalgebra $pMp$ is multiplicative. However, such projections $p$ do not exhaust the possibilities as the following observation shows.

Proposition 1.7. Let $p$ be an increasing projection in $M$ and let $\phi$ be the compression of $\alpha$ to $pMp$. Then $\phi$ is multiplicative iff $p$ commutes with $\alpha_t(pMp)$ for every $t \geq 0$.

Proof. Let $\phi_t(a) = p\alpha_t(a)p$, $a \in pMp$. If $p$ commutes with $\alpha_t(pMp)$ then it is clear that (1.4) is satisfied. Conversely, if (1.4) is satisfied then for every $a \in pMp$ we have

$$p\alpha_t(a^*)(1 - p)\alpha_t(a)p = p\alpha_t(a^*a)p - p\alpha_t(a^*)p\alpha_t(a)p = \phi_t(a^*a) - \phi_t(a^*)\phi_t(a) = 0,$$

and hence $(1 - p)\alpha_t(a)p = 0$. Thus the range of $p$ is invariant under the self-adjoint family of operators $\alpha_t(pMp)$, hence $p \in \alpha_t(pMp)'$. \qed

While the criterion of Proposition 1.7 is quite specific, it does not provide useful information for finding the multiplicative compressions of $\alpha$. Notice for example that the family of von Neumann algebras $\alpha_t(pMp)$ appearing there neither increases nor decreases with $t$, because while the projections $\alpha_t(p)$ increase with $t$ the von Neumann algebras $\alpha_t(M)$ decrease with $t$. In particular, Proposition 1.7 provides no insight into the order structure of the family of multiplicative compressions of $\alpha$. In section 3 we will prove the following two results concerning minimality.

Theorem A. Let $\alpha$ be an $E_0$-semigroup acting on a von Neumann algebra $M$ with separable predual, let $p$ be an increasing projection in $M$, and let $\phi$ be the compression of $\alpha$ to the hereditary subalgebra $pMp$.

There is an increasing projection $p_+ \geq p$ which defines a multiplicative compression of $\alpha$ such that if $q$ is any other increasing projection satisfying $q \geq p$ for which the compression of $\alpha$ to $qMq$ is multiplicative, then $q \geq p$. 
The compression of $\alpha$ to $p_+Mp_+$ defines an $E_0$-semigroup dilation of $\phi$ which is minimal over $\phi$.

**Remark.** Consider $M$ to be a subalgebra of $B(H)$ containing the identity operator. The following result identifies the subspace $p_+H$ in concrete terms and gives an algebraic criterion for minimality in the important case where $M$ is a factor.

**Theorem B.** Let $\alpha$ be an $E_0$-semigroup acting on a factor $M \subseteq B(H)$, $H$ being a separable Hilbert space. Let $p \in M$ be an increasing projection and let $\phi$ be the compression of $\alpha$ to $pMp$. The following are equivalent.

1. $\alpha$ is minimal over $\phi$.
2. $H$ is spanned by $$\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\cdots \alpha_{t_n}(a_n)\xi : a_1, \ldots, a_n \in pMp, t_k \geq 0, n \geq 1, \xi \in pH\}.$$ 3. $M$ is generated as a von Neumann algebra by the set of operators $$\{\alpha_t(a) : a \in pMp, t \geq 0\}.$$ 

2. **Projective cocycles.** Theorems A and B will be proved in section 3. They depend on properties of certain families of projections satisfying a cocycle equation.

**Definition 2.1.** Let $\alpha$ be an $E_0$-semigroup acting on $M$. A **projective cocycle** is a family of nonzero projections $\{p_t : t > 0\}$ in $M$ satisfying the following two conditions

\begin{align}
(2.1.1) & \quad p_t \in \alpha_t(M)' \\
(2.1.2) & \quad p_{s+t} = p_s \alpha_s(p_t), \quad s, t > 0.
\end{align}

Notice that we have imposed no regularity condition on the behavior of $p_t$ with respect to $t$. This will give us the flexibility we need for constructing examples. Nevertheless, projective cocycles are continuous:

**Proposition 2.2.** Let $p = \{p_t : t > 0\}$ be a projective cocycle. Then $p_t$ is a strongly continuous function of $t \in (0, \infty)$, and $p_t$ tends strongly to 1 as $t \to 0+$.

**proof.** The family of projections $p = \{p_t : t > 0\}$ determines a family of normal self-adjoint maps $\beta = \{\beta_t : t > 0\}$ of $M$ into itself by way of $$\beta_t(a) = p_t \alpha_t(a), \quad a \in M, t > 0.$$ 

Because of (2.1.1) each $\beta_t$ is an endomorphism of $M$, and (2.1.2) implies that $\beta$ has the semigroup property $\beta_{s+t} = \beta_s \beta_t$, $s, t > 0$. We have $\beta_t(1) = p_t$ for every $t > 0$.

Notice next that for fixed $\xi, \eta \in H$, the function $t \in (0, \infty) \mapsto \langle p_t \xi, \eta \rangle$ is Borel-measurable. Indeed, because of (2.1.1) and (2.1.2),

$$p_{s+t} = p_s \alpha_s(p_t) \leq p_s$$

for every $s, t > 0$ and hence $p_t$ is decreasing in $t$. It follows that for every $\xi \in H$, the function $$t \in (0, \infty) \mapsto \langle p_t \xi, \xi \rangle \in \mathbb{R}^+$$
is decreasing, therefore continuous except on a countable set, therefore measurable. The assertion about measurability of $t \mapsto \langle p_t \xi, \eta \rangle$ follows by polarization.

This implies that $\beta_t$ is weakly measurable in $t$ in the sense that for every $\xi, \eta \in H$ and every $a \in M$, the function $\langle \beta_t(a) \xi, \eta \rangle$ is Borel measurable. It follows that for every normal linear functional $\rho \in M^*$,

$$t \in (0, \infty) \mapsto \rho(\beta_t(a)) \in \mathbb{C}$$

is measurable. Since $M^*$ is separable we can apply Proposition 2.5 of [1] to conclude that for every $\rho \in M^*$ we have

$$\lim_{t \to 0^+} \|\rho \circ \beta_t - \rho\| = 0,$$

and

$$\lim_{t \to t_0} \|\rho \circ \beta_t - \rho \circ \beta_{t_0}\| = 0 \quad \text{for every } t_0 > 0.$$

In particular, taking $\rho(a) = \langle a \xi, \eta \rangle$ for fixed $\xi, \eta \in H$ we conclude that the function $\langle p_t \xi, \eta \rangle = \langle \beta_t(1) \xi, \eta \rangle$ is continuous in $t$ on the interval $(0, \infty)$ and that it tends to $\langle \xi, \eta \rangle = \langle 1 \xi, \eta \rangle$ as $t \to 0^+$.

The strong continuity of $\{p_t\}$ asserted in Prop. (2.2) follows because the strong and weak operator topologies coincide on the set of projections $\Box$

Thus, one may always assume that a projective cocycle $p = \{p_t : t \geq 0\}$ is defined on the entire nonnegative real axis, and satisfies the following two conditions in addition to the two properties of Definition 2.1:

(2.1.3) \hspace{1cm} p_0 = 1,

(2.1.4) \hspace{1cm} t \in [0, \infty) \mapsto p_t \text{ is strongly continuous}.

Remarks. Such cocycles have arisen in Powers’ recent work [12] on semigroups of endomorphisms of type I factors $M$. Given such a cocycle $p = \{p_t : t \geq 0\}$, one can form the associated semigroup of (nonunital) endomorphisms of $M$

$$\beta_t(a) = p_t \alpha_t(a).$$

Powers calls such a semigroup a compression of $\alpha$, and he shows that the set of all compressions of $\alpha$ is a conditionally complete lattice with respect to its natural ordering. The compressions of particular interest in [12] are the minimal ones, i.e., those of the form

$$\beta_t(a) = u_t a u_t^*$$

where $\{u_t : t \geq 0\}$ is a semigroup of isometries satisfying

$$\alpha_t(a) u_t = u_t a, \quad a \in M, t \geq 0.$$

Notice that in this case the projective cocycle $p$ is related to the semigroup $u$ by $p_t = u_t u_t^*, t \geq 0$.

We will not use Powers’ terminology in this paper because we are concerned with dilation theory, and in dilation theory the term compression carries a somewhat broader meaning. Moreover, our need for projective cocycles has grown from considerations that are quite different from those of [12], and it will be more convenient for us to deal directly with the cocycles rather than with their associated semigroups.
Definition 2.3. Let \( p = \{p_t : t > 0\} \) and \( q = \{q_t : t > 0\} \) be two projective cocycles. We write \( p \leq q \) if \( p_t \leq q_t \) for every \( t > 0 \).

The following proposition gives a general procedure for constructing projective cocycles in arbitrary von Neumann algebras from families of projections having somewhat less structure.

**Proposition 2.4.** Let \( \alpha \) be an \( E_0 \)-semigroup acting on a von Neumann algebra \( M \) and let \( \{f_t : t > 0\} \) be a family of nonzero projections in \( M \) satisfying

\[
\begin{align*}
(2.4.1) & \quad f_t \in \alpha_t(M)' \\
(2.4.2) & \quad f_{s+t} \leq f_s \alpha_s(f_t), \quad s, t > 0.
\end{align*}
\]

Fix \( t > 0 \) and consider the set of all finite partitions

\[
\mathcal{P} = \{0 < t_1 < t_2 < \cdots < t_n = t\}
\]

of the interval \([0, t]\) as an increasing directed set in the usual sense. For such a partition \( \mathcal{P} \), define an operator \( f_\mathcal{P} \) by

\[
f_\mathcal{P} = f_{t_1} \alpha_{t_1}(f_{t_2-t_1}) \alpha_{t_2}(f_{t_3-t_2}) \cdots \alpha_{t_{n-1}}(f_{t_n-t_{n-1}}).
\]

\( f_\mathcal{P} \) is a projection and \( \mathcal{P}_1 \subseteq \mathcal{P}_2 \implies f_{\mathcal{P}_1} \leq f_{\mathcal{P}_2} \). Thus we can define a projection \( p_t \) by

\[
p_t = \sup_{\mathcal{P}} f_\mathcal{P} = \lim_{\mathcal{P}} f_\mathcal{P}.
\]

The family \( p = \{p_t : t > 0\} \) is a projective cocycle, and is the smallest projective cocycle \( p \) such that \( f_t \leq p_t \) for every \( t > 0 \).

**proof.** Let \( s, t > 0 \) and let \( a \) and \( b \) be operators in \( M \) such that \( a \) commutes with \( \alpha_s(M) \) and \( b \) commutes with \( \alpha_t(M) \). Then \( a \) commutes with \( \alpha_s(b) \) and note that the product \( a \alpha_s(b) \) commutes with \( \alpha_{s+t}(M) \). Indeed, for arbitrary \( c \in M \) we have

\[
aa_s(b)\alpha_{s+t}(c) = aa_s(b\alpha_t(c)) = \alpha_s(\alpha_t(c)b)a = \alpha_{s+t}(c)\alpha_s(b)a = \alpha_{s+t}(c)a\alpha_s(b).
\]

Now fix \( t > 0 \). It follows from the preceding remarks that the operator \( f_\mathcal{P} \) of (2.4.3) belongs to \( M \cap \alpha_t(M)' \); moreover, the \( n \) factors of \( f_\mathcal{P} \) on the right side of (2.4.3) are mutually commuting projections. Thus \( f_\mathcal{P} \) is a projection in \( M \cap \alpha_t(M)' \).

To show that \( f_\mathcal{P} \) increases with \( \mathcal{P} \) it is enough to show that if a given partition \( \mathcal{P} = \{0 < t_1 < \cdots < t_n = t\} \) is refined by adjoining to it a single point \( \tau \), then \( f_\mathcal{P} \) increases. In turn, that reduces to the following assertion. For \( k = 1, 2, \ldots, n \) and \( t_{k-1} < \tau < t_k \) (where \( t_0 \) is taken as 0),

\[
f_{t_{k-1}} \leq f_{\tau-t_{k-1}} \alpha_{\tau-t_{k-1}}(f_{t_k-\tau}).
\]

The latter is immediate from the hypothesis (2.4.2).

Thus the net \( f_\mathcal{P} \) increases with \( \mathcal{P} \) and we can define a projection \( p_t \in M \cap \alpha_t(M)' \) as asserted. The cocycle property (2.1.2) follows immediately from the definition of the family \( \{p_t : t > 0\} \).

We obviously have \( f_t \leq p_t \) for every \( t > 0 \). Finally, suppose \( q = \{q_t : t > 0\} \) is another projective cocycle satisfying \( f_t \leq q_t \) for every \( t > 0 \). Fix \( t > 0 \) and let
\[ P = \{ 0 < t_1 < \cdots < t_n = t \} \] be a partition of the interval \([0, t]\). Then for every \(k = 1, 2, \ldots, n\) we have \(f_{t_k-t_{k-1}} \leq q_{t_k-t_{k-1}}\), and hence \(f_P \leq q_P\). On the other hand, the cocycle property of \(q\) implies that \(q_P = q_t\). Hence \(f_P \leq q_t\) and we deduce the desired inequality

\[ p_t = \sup_P f_P \leq q_t. \]

\[ \square \]

The following result will be important for section 3. It implies that certain projections in \(M\) naturally give rise to projective cocycles.

**Corollary 2.5.** Let \(e\) be a nonzero projection in \(M\) satisfying \(\alpha_t(e) \geq e\) for every \(t \geq 0\). For each \(t > 0\) let \(f_t\) be the smallest projection in \(M \cap \alpha_t(M)'\) which dominates \(e\), i.e.,

\[ f_t = [\alpha_t(a)e_\xi : a \in M, \xi \in H]. \]

Then \(f_{s+t} \leq f_s \alpha_s(f_t)\) for every \(s, t > 0\). The projective cocycle \(p = \{p_t : t > 0\}\) of Proposition 2.4 is the smallest projective cocycle satisfying \(p_t \geq e\) for every \(t > 0\).

**proof.** It is obvious that \(f_t\) commutes with \(\alpha_t(M)\), and the double commutant theorem implies that \(f_t \in M\). Hence \(f_t \in M \cap \alpha_t(M)\).

To see that \(f_{s+t} \leq f_s \alpha_s(f_t)\), fix \(s, t > 0\). By the argument at the beginning of the proof of proposition 2.4, \(f_s \alpha_s(f_t)\) is a projection in \(M \cap \alpha_{s+t}(M)'\). We claim that \(e \leq f_s \alpha_s(f_t)\). Indeed, \(e \leq f_s\) follows from the definition of \(f_s\), and since \(e \leq f_t\) implies \(\alpha_s(e) \leq \alpha_s(f_t)\) we have \(e \leq \alpha_s(e) \leq \alpha_s(f_t)\). Hence \(e \leq f_s \alpha_s(f_t)\). Since \(f_{s+t}\) is the smallest projection in \(M \cap \alpha_{s+t}(M)\) which dominates \(e\) we have the asserted inequality \(f_{s+t} \leq f_s \alpha_s(f_t)\) \(\square\)

### 3. Minimality.

Throughout the section, \(\alpha\) will denote an \(E_0\)-semigroup acting on a von Neumann algebra \(M \subseteq B(H)\) and \(p\) will denote a projection in \(M\) satisfying

\[ (3.1) \quad \alpha_t(p) \geq p, \quad t \geq 0. \]

\(M_0\) will denote the hereditary subalgebra \(pMp\). We will be concerned with the (perhaps nonunital) von Neumann algebra \(M_+\) generated by \(M_0\) and its translates under \(\alpha\),

\[ M_+ = \overline{\text{span}}\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2) \cdots \alpha_{t_n}(a_n) : a_k \in M_0, t_k \geq 0, n = 1, 2, \ldots\}, \]

the bar denoting closure in the weak operator topology. It is obvious that \(\alpha_t(M_+) \subseteq M_+\) for every \(t \geq 0\), and the unit of \(M_+\) is the projection

\[ (3.3) \quad p_\infty = \lim_{t \to \infty} \alpha_t(p). \]

There is a smaller projection that is of greater importance, namely

\[ (3.4) \quad p_+ = [M_+pH] = [\alpha_{t_1}(a_1)\alpha_{t_2}(a_2) \cdots \alpha_{t_n}(a_n)\xi : a_k \in M_0, t_k \geq 0, \xi \in pH]. \]

**Remarks.** \(p_+\) is the unit of the two-sided ideal \(\overline{\text{span}}M_+pM_+\) of \(M_+\) generated by \(p\). Thus, \(p_+\) belongs to the center of \(M_+\), and in fact is the smallest central projection \(c\) in \(M_+\) satisfying \(p \leq c\).

We will eventually show that \(p_+\) is an increasing projection which defines a multiplicative compression of \(\alpha\). Neither of these assertions is apparent from (3.4). We deduce these properties from the following result which gives a “formula” for \(p_+\).
Theorem 3.5. Let $p$ be a projection in $M$ which satisfies (3.1), let $p_\infty$ and $p_+$ be defined as in (3.3) and (3.4) respectively. Let $q = \{q_t : t > 0\}$ be the smallest projective cocycle satisfying $q_t \geq p$ for every $t \geq 0$ as in Corollary 2.5, and set
\[ q_\infty = \lim_{t \to \infty} q_t. \]

Then $p_\infty$ belongs to the tail von Neumann algebra $M_\infty = \cap_{t \geq 0} \alpha_t(M)$, $q_\infty$ belongs to its relative commutant in $M$, and we have a factorization
\[ p_+ = p_\infty q_\infty. \]

Remarks. Recall that since $q$ is a projective cocycle, $q_t$ must be a decreasing function of $t$ and hence the strong limit $q_\infty = \lim_{t \to \infty} q_t$ exists.

proof. Notice first that since $\alpha_t(p) \subseteq \alpha_t(M)$ and since the von Neumann algebras $\alpha_t(M)$ decrease as $t$ increases, it follows that $p_\infty = \lim_{t \to \infty} p_t \subseteq M_\infty$. Since $q_t$ belongs to $M \cap \alpha_t(M)' \subseteq M \cap M_\infty'$ for all $t$ we see that $q_\infty = \lim_{t \to \infty} q_t \in M \cap M_\infty'$. In particular, the projections $p_\infty$ and $q_\infty$ must commute.

We show first that $p_+ \leq p_\infty q_\infty$. Since $p_+ \leq p_\infty$ is obvious, it suffices to show that $p_+ H \leq q_\infty H$. Considering the definition of $p_+$ and the fact that $pH \leq q_\infty H$, it suffices to show that the subspace $q_\infty H$ is invariant under any operator in any one of the von Neumann algebras $\alpha_t(M_0)$, $t > 0$, i.e., that $q_\infty$ commutes with $\cup_{t > 0} \alpha_t(M_0)$. For that, fix $t > 0$. If we pass $s$ to $\infty$ in the cocycle formula
\[ q_t \alpha_t(q_s) = q_{t+s} \]
and use normality of $\alpha_t$ we obtain
\[ q_t \alpha_t(q_\infty) = q_\infty. \]

It follows that for any $a \in M_0$ we have
\[ \alpha_t(a) q_\infty = \alpha_t(a) q_t \alpha_t(q_\infty) = q_t \alpha_t(a) \alpha_t(q_\infty) = q_t \alpha_t(a q_\infty), \]
where we have used $q_t \in \alpha_t(M)'$. Now since $p \leq q_\infty$ and since $a = p a p$ we have $a q_\infty = a q_\infty a$. Thus we can replace the right side of (3.7) with
\[ q_t \alpha_t(q_\infty a) = q_t \alpha_t(q_\infty) \alpha_t(a) = q_\infty \alpha_t(a). \]

Thus $\alpha_t(a)$ commutes with $q_\infty$ as required.

It remains to show that $p_\infty q_\infty \leq p_+$. For that, it suffices to show that for every $t > 0$ we have
\[ q_t \alpha_t(p_+) \leq p_. \]

Indeed, assuming that (3.8) has been established we deduce $q_t \alpha_t(p) \leq p_+$ for every $t$ (because $p \leq p_+$); noting that $q_t \downarrow q_\infty$ and $\alpha_t(p) \uparrow p_\infty$ as $t$ increases to $+\infty$, we may take the strong limit on $t$ in the previous formula to obtain the desired inequality $q_\infty p_\infty \leq p_+$.

In order to prove (3.8), we require
Lemma 3.9. For each \( t > 0 \) let \( f_t \) be the projection onto the subspace 
\[ \{ \alpha_t(a)p\xi : a \in M, \xi \in H \} \].

Then \( f_t \alpha_t(p+) \leq p_+ \).

\textbf{proof.} We have already pointed out in the remarks following (3.4) that \( p_+ \) is the unit of the ideal \( \text{span} M_+pM_+ \) in \( M_+ \). Thus it suffices to show that 
\[ f_t \alpha_t(M_+pM_+)H \subseteq p_+H. \]

Since \( f_t \) commutes with \( \alpha_t(M) \) the left side is contained in 
\[ \alpha_t(M_+pM_+)f_tH \subseteq [\alpha_t(M_+pM_+)\alpha_t(M)pH] \subseteq [\alpha_t(M_+pM_+)pH]. \]

Noting that \( p = \alpha_t(p)p \) the latter is 
\[ [\alpha_t(M_+pM_+MpH)] \subseteq [\alpha_t(M_+pMpH)] \subseteq [\alpha_t(M_+)pH] \subseteq [M_+pH] = p_+H, \]

as asserted \( \Box \)

For every \( t > 0 \) we define a normal linear mapping \( \beta_t : M \to M \) as follows, 
\[ \beta_t(a) = f_t \alpha_t(a). \]

Since \( f_t \) commutes with \( \alpha_t(M) \), \( \beta_t \) is an endomorphism of the *-algebra structure of \( M \) for which \( \beta_t(1) = f_t \), but it is not a semigroup because \( \{ f_t : t > 0 \} \) does not satisfy the cocycle condition (1.1.2). However, because of Lemma 3.9 we have 
\[ \beta_t(p+) \leq p_+, \quad t > 0. \]

Now fix \( t > 0 \) and let \( \mathcal{P} = \{ 0 = t_0 < t_1 < \cdots < t_n = t \} \) be a partition of the interval \([0,t]\). By iterating the preceding formula we find that 
\[ \beta_{t_1} \beta_{t_2-t_1} \beta_{t_3-t_2} \cdots \beta_{t_n-t_{n-1}}(p_+) \leq p_+. \]

In the notation of Proposition 1.4, the left side of (3.10) is 
\[ f_{t_1} \alpha_{t_1} (f_{t_2-t_1}) \alpha_{t_2} (f_{t_3-t_2}) \cdots \alpha_{t_{n-1}} (f_{t_n-t_{n-1}}) \alpha_t(p+) = f_\mathcal{P} \alpha_t(p_+). \]

Using 1.4 and 1.5, we make take the limit on \( \mathcal{P} \) in (3.10) obtain the required inequality (3.8) 
\[ q_t \alpha_t(p+) = \lim_{\mathcal{P}} f_\mathcal{P} \alpha_t(p+) \leq p_+, \]

completing the proof of Theorem 3.5 \( \Box \)

We can now deduce the following result, which paraphrases Theorem A from section 1.
Theorem A. Let $p$ be an increasing projection for $\alpha$ and let $p_+ \geq p$ be the projection defined by (3.4). $p_+$ is an increasing projection with the property that the compression of $\alpha$ to $p_+ M p_+$ is multiplicative.

If $r$ is another increasing projection in $M$ such that $r \geq p$ and the compression of $\alpha$ to $r M r$ is multiplicative, then $r \geq p_+$.

proof. Since $p_\infty = \lim_{t \to -\infty} \alpha_t(p)$ is clearly fixed under the action of $\alpha_t$ and since (3.6) implies that $\alpha_t(q_\infty) \geq q_\infty$, we find that $\alpha_t(p_+) = \alpha_t(p_\infty) \alpha_t(q_\infty) \geq p_\infty q_\infty = p_+$.

To show that the compression of $\alpha$ to $p_+ M p_+$ is multiplicative, it suffices to show that $p_+$ commutes with $\alpha_t(p_+ M p_+)$ for every $t > 0$ (Proposition 1.7). For that, it is enough to show that for every $a = a^* \in p_+ M p_+$ we have

\begin{equation}
(3.11) \quad p_+ \alpha_t(a) = q_t \alpha_t(a).
\end{equation}

Indeed, by taking adjoints in (3.11) we find that $\alpha_t(a) q_t = \alpha_t(a) p_+$, and since $q$ is a projective cocycle $q_t$ must commute with $\alpha_t(M)$. Thus

\[ p_+ \alpha_t(a) = q_t \alpha_t(a) = \alpha_t(a) q_t = \alpha_t(a) p_+, \]

and thus $p_+ \in \alpha_t(p_+ M p_+)'$.

To prove (3.11), we write $p_+ = p_\infty q_\infty = q_\infty p_\infty$ and use $\alpha_t(p_\infty) = p_\infty$ to obtain

\begin{equation}
(3.12) \quad p_+ \alpha_t(a) = q_\infty p_\infty \alpha(t(a) = q_\infty \alpha_t(p_\infty a) = q_\infty \alpha_t(a),
\end{equation}

because $p_\infty a = a$ for every operator $a$ in $p_+ M p_+ \subseteq p_\infty M p_\infty$. Using (3.6) on the last term of (3.12) we have

\[ q_\infty \alpha_t(a) = q_t \alpha_t(p_\infty) \alpha_t(a) = q_t \alpha_t(q_\infty a) = q_t \alpha_t(a), \]

since $q_\infty a = a$ for every $a \in p_+ M p_+ \subseteq q_\infty M q_\infty$. Formula (3.11) follows.

Finally, let $r \geq p$ be another increasing projection with the property that the compression of $\alpha$ to $r M r$ is multiplicative. We have to show that $p_+ H \subseteq r H$. Because of formula (3.4) for $p_+ H$, together with the fact that $p H \subseteq r H$, it is enough to show that $r H$ is invariant under any operator of the form $\alpha_t(a)$ with $a \in p M p$ and $t > 0$. But for each $t > 0$, Proposition 1.7 implies that $r$ commutes with the set of operators $\alpha_t(r M r)$, and therefore since $p \leq r$ we have

\[ \alpha_t(p M p) r H \subseteq \alpha_t(r M r) r H \subseteq r H, \]

as required \(\square\)

As another consequence of Theorem 3.5 we have the following characterization of minimality in terms of projective cocycles.

Corollary 3.13. Let $p$ be an increasing projection and let $\phi$ be the compression of $\alpha$ to $p M p$. Then $\alpha$ is minimal over $\phi$ iff $\lim_{t \to -\infty} \alpha_t(p) = 1$ and the only projective cocycle $q = \{ q_t : t > 0 \}$ satisfying $q_t \geq p$ for every $t > 0$ is the trivial cocycle $q_t = 1$.

proof. The minimality assertion is that $p_+ = 1$ and from Theorem 3.5 we have $p_+ = p_\infty q_\infty$. Thus $\alpha$ is minimal iff $p_\infty = q_\infty = 1$ \(\square\)
Proposition 3.14. Let $M_+$ be the following von Neumann subalgebra of $M$

$$M_+ = \overline{\text{span}}\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\ldots\alpha_{t_n}(a_n) : a_1,\ldots,a_n \in pMp, t_1,\ldots,t_n \geq 0, n \geq 1\},$$

and let $p_+$ be the projection of (3.4). Then $p_+$ is the smallest projection in the center of $M_+$ which dominates $p$, and we have $p_+Mp_+ = M_+p_+$.

proof. While the first assertion is very elementary, we include a proof for completeness. Clearly $p_+ \in M_+$, and since $p_+H = [M_+pH]$ is invariant under $M_+$ we have $p_+ \in M_+'$. Hence $p_+$ belongs to the center of $M_+$ and $p \leq p_+$. If $c$ is another central projection in $M_+$ for which $c \geq p$, then $cH$ clearly contains $[M_+pH] = p_+H$ and hence $c \geq p_+$.

Let $R$ be the weakly closed subspace of $M$ generated by the set of operators

$$\{apb : a \in M_+, b \in M\}.$$ 

$R$ is a right ideal in $M$, and the range projection of $R$ is

$$[RH] = [apH : a \in M_+] = p_+H.$$ 

Hence $R = p_+M$. Thus $p_+Mp_+$ is spanned by $RR^*$, i.e.,

$$p_+Mp_+ = \overline{\text{span}}(M_+ \cdot pMp \cdot M_+) = \overline{\text{span}}(M_+pM_+).$$

The right side of the preceding formula is the two-sided ideal in $M_+$ generated by $p$ which, by the preceding paragraph, is $M_+p_+$. □

Theorem B. Let $\alpha$ be an $E_0$-semigroup acting on a factor $M \subseteq B(H)$, $H$ being a separable Hilbert space. Let $p \in M$ be an increasing projection and let $\phi$ be the compression of $\alpha$ to $pMp$. The following are equivalent.

1. $\alpha$ is minimal over $\phi$.
2. $H$ is spanned by

$$\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\ldots\alpha_{t_n}(a_n)\xi : a_1,\ldots,a_n \in pMp, t_k \geq 0, n \geq 1, \xi \in pH\}.$$ 

3. $M$ is generated as a von Neumann algebra by the set of operators

$$\{\alpha_t(a) : a \in pMp, t \geq 0\}.$$ 

proof of (1) $\implies$ (3). If $\alpha$ is minimal over $\phi$ then $p_+ = 1$, and thus by Proposition 3.14 we find that $M = p_+Mp_+ = M_+p_+ = M_+$, hence (3).

proof of (3) $\implies$ (2). Since $M = M_+$ we have $[M_+pH] = [MpH]$. The projection on the subspace on the right is the central carrier of $p$, which must be 1 because $M$ is a factor. Therefore $p_+H = H$, as asserted in (2).

proof of (2) $\implies$ (1). The hypothesis is that $p_+ = 1$ which, by Theorem A, implies that $\alpha$ is minimal over $\phi$. □
References