MINIMAL E_0-SEMIGROUPS

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Abstract. It is known that every semigroup of normal completely positive maps of a von Neumann can be "dilated" in a particular way to an E_0-semigroup acting on a larger von Neumann algebra. The E_0-semigroup is not uniquely determined by the completely positive semigroup; however, it is unique (up to conjugacy) provided that certain conditions of *minimality* are met. Minimality is a subtle property, and it is often not obvious if it is satisfied for specific examples even in the simplest case where the von Neumann algebra is $\mathcal{B}(H)$.

In this paper we clarify these issues by giving a new characterization of minimality in terms of projective cocycles and their limits. Our results are valid for semigroups of endomorphisms acting on arbitrary von Neumann algebras with separable predual.

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1. Dilations and compressions. Let M be a von Neumann algebra with separable predual. When it is convenient to do so, we will consider that M is a von Neumann subalgebra of the algebra $\mathcal{B}(H)$ of all bounded operators on a separable Hilbert space H which contains the identity operator $\mathbf{1}$. The separability of H will be essential for some of the results below.

By an E_0-semigroup acting on M we mean a family of normal $*$-endomorphisms $\alpha = \{\alpha_t : t \geq 0\}$ of M satisfying $\alpha_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$, which obeys the semigroup property $\alpha_{s+t} = \alpha_s \alpha_t$, and which is continuous in the sense that for every $a \in M$ and every pair of vectors $\xi, \eta \in H$, the function $t \in [0, \infty) \mapsto \langle \alpha_t(a)\xi, \eta \rangle$ is continuous. We will also consider semigroups $\phi = \{\phi_t : t \geq 0\}$ of normal completely positive maps acting on certain von Neumann subalgebras $N \subseteq M$. These subalgebras will normally not contain the unit of M; but we will require that such semigroups ϕ be unital in the sense that $\phi_t(\mathbf{1}_N) = \mathbf{1}_N$, $t \geq 0$, and that they should satisfy the natural continuity property cited above. We will refer to such a semigroup $\phi_t : N \to N$ simply as a *completely positive semigroup*.

α can be compressed to *certain* hereditary subalgebras of M so as to give a completely positive semigroup as follows. Let $M_0 = pMp$ be a hereditary von Neumann subalgebra of M with unit p. The natural projection

$$E_0 : M \to M_0$$

of M onto M_0 is defined by $E_0(a) = pap$. E_0 carries the unit of M to that of M_0, and we have

$$E_0(axb) = aE_0(x)b, \qquad a, b \in M_0, x \in M.$$

Fix $t \geq 0$. It is an elementary exercise to show that in order for there to exist a linear map $\phi_t : M_0 \to M_0$ satisfying $E_0 \circ \alpha_t = \phi_t \circ E_0$ it is necessary and sufficient that $\alpha_t(p) \geq p$. Thus we will be concerned with hereditary subalgebras $M_0 = pMp$ for which the projection p is *increasing* in the sense that

$$\alpha_t(p) \geq p, \qquad t \geq 0. \tag{1.1}$$

In this case one can define a family $\phi = \{\phi_t : t \geq 0\}$ of completely positive maps of M_0 by compressing each map α_t to M_0,

$$\phi_t(a) = p\alpha_t(a)p, \qquad a \in M_0, t \geq 0. \tag{1.2}$$

Since $\phi_t(p) = p$, we may consider ϕ_t to be a unital map of M_0. Note too that the family ϕ has the semigroup property $\phi_{s+t} = \phi_s \phi_t$ for $s, t \geq 0$. Indeed, since $p\alpha_s(p) = \alpha_s(p)p = p$ by (1.1) we have

$$\phi_s\phi_t(a) = p\alpha_s(p\alpha_t(a)p)p = p\alpha_s(p)\alpha_{s+t}(a)\alpha_s(p)p = \phi_{s+t}(a),$$

for every $a \in M_0$. Thus ϕ is a completely positive semigroup acting on M_0. Throughout this paper we will be concerned with properties of completely positive semigroups which can be obtained from a fixed E_0-semigroup in this particular way.

Definition 1.3. *Let p be a projection in M satisfying (1.1), and let pMp be the corresponding hereditary subalgebra. The completely positive semigroup $\phi = \{\phi_t : t \geq 0\}$ defined on pMp by (1.2) is called a* **compression** *of α, and α is called a* **dilation** *of ϕ.*

We emphasize that the notion of a compression of α to a subalgebra has meaning only when the subalgebra is (1) a hereditary subalgebra pMp of M and (2) p is a projection satisfying (1.1). It is possible to conjure up other definitions of "compressions" of α and "dilations" of ϕ. For example, one might imagine using a conditional expectation from M onto a *unital* subalgebra N of M to attempt to define a semigroup of completely positive maps of N under appropriate conditions. While there has been some limited success for such endeavors [8],[9], a recent theorem of B. V. R. Bhat [5] has led us to the conclusion that the proper context for this kind of dilation theory is the context of Definition 1.3.

More precisely, Bhat has shown that for every completely positive semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on a von Neumann algebra M_0, there is a larger von Neumann algebra M containing M_0 as a *hereditary* subalgebra $M_0 = pMp$ and an E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on M which satisfies $\alpha_t(p) \geq p$ for every t and is such that ϕ is obtained from α by compression. Thus we have a dilation theory which resembles the more familiar dilation theory for operator semigroups which asserts that every semigroup of contraction operators on a Hilbert space can be dilated to a semigroup of isometries on a larger Hilbert space.

In the case of operator semigroups there is a simple notion of *minimal* isometric dilation, and two minimal isometric dilations of the same contraction semigroup are naturally unitarily equivalent. There is an analogous notion of minimality in the current setting and there is an analogous uniqueness result for the minimal E_0-semigroup dilation of a completely positive semigroup [5], [6]. However, these considerations for completely positive semigroups and their E_0-semigroup dilations are much more subtle than their counterparts in operator theory. To illustrate the level of subtlety, recall that a semigroup of contraction operators can actually be dilated further to a semigroup of unitary operators. That is because any semigroup of isometries acting on a Hilbert space is the restriction to an invariant subspace of a semigroup of unitary operators acting on a larger Hilbert space. The unitary semigroup is unique (up to a natural unitary equivalence) if the invariant subspace is "minimal". Nothing like that is true in this setting. Indeed, an E_0-semigroup acting on a type I factor M does not have a *natural* extension to a semigroup of automorphisms of a larger type I factor which contains M as a unital subfactor. It is true that every E_0-semigroup can be so extended, but the construction of the extension is quite indirect and there is apparently no uniqueness of such extensions (see [2], or see [4] for a different proof). Finally, the notion of minimality introduced in [5], [6] is geared to quantum probability theory and does not lend itself readily to the natural questions that arise in the theory of E_0-semigroups.

The purpose of this paper is to clarify the issue of minimality for E_0-semigroups acting on arbitrary von Neumann algebras, and to give a new characterization of minimality in terms of the natural objects of operator algebras.

Let p be an increasing projection in M and let $\phi = \{\phi_t : t \geq 0\}$ be the compression of α to pMp. Notice that ϕ is itself an E_0-semigroup (acting on pMp) iff for every $t \geq 0$ we have

$$\phi_t(ab) = \phi_t(a)\phi_t(b), \qquad a, b \in pMp. \tag{1.4}$$

Definition 1.5. *Let α be an E_0-semigroup acting on a von Neumann algebra M. A compression ϕ of α to a hereditary subalgebra which satisfies property (1.4) is called* **multiplicative***.*

Suppose that ϕ is a compression of an E_0-semigroup α to a hereditary subalgebra pMp of M, and that q is an increasing projection such that $q \geq p$ and the compression of α to qMq is *multiplicative*. Then may consider that the compression of α to the intermediate subalgebra qMq is itself an E_0-semigroup which has ϕ as a compression.

Definition 1.6. *Let α be an E_0-semigroup acting on a von Neumann algebra M, let p be an increasing projection in M, and let ϕ be the completely positive semigroup on pMp obtained by compression. α is said to be* **minimal** *over ϕ if the only increasing projection $q \in M$ which satisfies $q \geq p$, and is such that the comression of α to qMq is multiplicative, is the projection $q = \mathbf{1}$.*

In order to discuss minimality further, one needs to know more about increasing projections which define multiplicative compressions. Here is the simplest class of examples. Let p be a projection of M which is *fixed* under α in the sense that

$$\alpha_t(p) = p, \qquad t \geq 0.$$

In this case it is clear that the compression of α to the hereditary subalgebra pMp is multiplicative. However, such projections p do not exhaust the possibilities as the following observation shows.

Proposition 1.7. *Let p be an increasing projection in M and let ϕ be the compression of α to pMp. Then ϕ is multiplicative iff p commutes with $\alpha_t(pMp)$ for every $t \geq 0$.*

proof. Let $\phi_t(a) = p\alpha_t(a)p$, $a \in pMp$. If p commutes with $\alpha_t(pMp)$ then it is clear that (1.4) is satisfied. Conversely, if (1.4) is satisfied then for every $a \in pMp$ we have

$$p\alpha_t(a)^*(\mathbf{1}-p)\alpha_t(a)p = p\alpha_t(a^*a)p - p\alpha_t(a^*)p\alpha_t(a)p = \phi_t(a^*a) - \phi_t(a^*)\phi_t(a) = 0,$$

and hence $(\mathbf{1}-p)\alpha_t(a)p = 0$. Thus the range of p is invariant under the self-adjoint family of operators $\alpha_t(pMp)$, hence $p \in \alpha_t(pMp)'$ □

While the criterion of Proposition 1.7 is quite specific, it does not provide useful information for finding the multiplicative compressions of α. Notice for example that the family of von Neumann algebras $\alpha_t(pMp)$ appearing there neither increases nor decreases with t, because while the projections $\alpha_t(p)$ increase with t the von Neumann algebras $\alpha_t(M)$ *decrease* with t. In particular, Proposition 1.7 provides no insight into the order structure of the family of multiplicative compressions of α. In section 3 we will prove the following two results concerning minimality.

Theorem A. *Let α be an E_0-semigroup acting on a von Neumann algebra M with separable predual, let p be an increasing projection in M, and let ϕ be the compression of α to the hereditary subalgebra pMp.*

There is an increasing projection $p_+ \geq p$ which defines a multiplicative compression of α such that if q is any other increasing projection satisfying $q \geq p$ for which the compression of α to qMq is multiplicative, then $q \geq p_+$.

The compression of α to p_+Mp_+ defines an E_0-semigroup dilation of ϕ which is minimal over ϕ.

Remark. Consider M to be a subalgebra of $\mathcal{B}(H)$ containing the identity operator. The following result identifies the subspace p_+H in concrete terms and gives an algebraic criterion for minimality in the important case where M is a factor.

Theorem B. *Let α be an E_0-semigroup acting on a factor $M \subseteq \mathcal{B}(H)$, H being a separable Hilbert space. Let $p \in M$ be an increasing projection and let ϕ be the compression of α to pMp. The following are equivalent.*

(1) *α is minimal over ϕ.*
(2) *H is spanned by*

$$\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\dots\alpha_{t_n}(a_n)\xi : a_1,\dots,a_n \in pMp, t_k \geq 0, n \geq 1, \xi \in pH\}.$$

(3) *M is generated as a von Neumann algebra by the set of operators*

$$\{\alpha_t(a) : a \in pMp, t \geq 0\}.$$

2. Projective cocycles. Theorems A and B will be proved in section 3. They depend on properties of certain families of projections satisfying a cocycle equation.

Definition 2.1. *Let α be an E_0-semigroup acting on M. A* **projective cocycle** *is a family of nonzero projections $\{p_t : t > 0\}$ in M satisfying the following two conditions*

$$p_t \in \alpha_t(M)' \tag{2.1.1}$$

$$p_{s+t} = p_s\alpha_s(p_t), \qquad s,t > 0. \tag{2.1.2}$$

Notice that we have imposed no regularity condition on the behavior of p_t with respect to t. This will give us the flexibility we need for constructing examples. Nevertheless, projective cocycles are continuous:

Proposition 2.2. *Let $p = \{p_t : t > 0\}$ be a projective cocycle. Then p_t is a strongly continuous function of $t \in (0,\infty)$, and p_t tends strongly to $\mathbf{1}$ as $t \to 0+$.*

proof. The family of projections $p = \{p_t : t > 0\}$ determines a family of normal self-adjoint maps $\beta = \{\beta_t : t > 0\}$ of M into itself by way of

$$\beta_t(a) = p_t\alpha_t(a), \qquad a \in M, t > 0.$$

Because of (2.1.1) each β_t is an endomorphism of M, and (2.1.2) implies that β has the semigroup property $\beta_{s+t} = \beta_s\beta_t$, $s,t > 0$. We have $\beta_t(\mathbf{1}) = p_t$ for every $t > 0$.

Notice next that for fixed $\xi, \eta \in H$, the function $t \in (0,\infty) \mapsto \langle p_t\xi, \eta\rangle$ is Borel-measurable. Indeed, because of (2.1.1) and (2.1.2),

$$p_{s+t} = p_s\alpha_s(p_t) \leq p_s$$

for every $s,t > 0$ and hence p_t is decreasing in t. It follows that for every $\xi \in H$, the function

$$t \in (0,\infty) \mapsto \langle p_t\xi, \xi\rangle \in \mathbb{R}^+$$

is decreasing, therefore continuous except on a countable set, therefore measurable. The assertion about measurability of $t \mapsto \langle p_t\xi, \eta\rangle$ follows by polarization.

This implies that β_t is weakly measurable in t in the sense that for every $\xi, \eta \in H$ and every $a \in M$, the function $\langle \beta_t(a)\xi, \eta\rangle$ is Borel measurable. It follows that for every normal linear functional $\rho \in M_*$,

$$t \in (0, \infty) \mapsto \rho(\beta_t(a)) \in \mathbb{C}$$

is measurable. Since M_* is separable we can apply Proposition 2.5 of [1] to conclude that for every $\rho \in M_*$ we have

$$\lim_{t\to 0+} \|\rho \circ \beta_t - \rho\| = 0,$$

and

$$\lim_{t\to t_0} \|\rho \circ \beta_t - \rho \circ \beta_{t_0}\| = 0 \qquad \text{for every } t_0 > 0.$$

In particular, taking $\rho(a) = \langle a\xi, \eta\rangle$ for fixed $\xi, \eta \in H$ we conclude that the function $\langle p_t\xi, \eta\rangle = \langle \beta_t(\mathbf{1})\xi, \eta\rangle$ is continuous in t on the interval $(0, \infty)$ and that it tends to $\langle \xi, \eta\rangle = \langle \mathbf{1}\xi, \eta\rangle$ as $t \to 0+$.

The strong continuity of $\{p_t\}$ asserted in Prop. (2.2) follows because the strong and weak operator topologies coincide on the set of projections $\square$

Thus, one may always assume that a projective cocycle $p = \{p_t : t \geq 0\}$ is defined on the entire nonnegative real axis, and satisfies the following two conditions in addition to the two properties of Definition 2.1:

$$p_0 = \mathbf{1}, \tag{2.1.3}$$

$$t \in [0, \infty) \mapsto p_t \text{ is strongly continuous.} \tag{2.1.4}$$

Remarks. Such cocycles have arisen in Powers' recent work [12] on semigroups of endomorphisms of type I factors M. Given such a cocycle $p = \{p_t : t \geq 0\}$, one can form the associated semigroup of (nonunital) endomorphisms of M

$$\beta_t(a) = p_t\alpha_t(a).$$

Powers calls such a semigroup a *compression* of α, and he shows that the set of all compressions of α is a conditionally complete lattice with respect to its natural ordering. The compressions of particular interest in [12] are the *minimal* ones, i.e., those of the form

$$\beta_t(a) = u_t a u_t^*$$

where $\{u_t : t \geq 0\}$ is a semigroup of isometries satisfying

$$\alpha_t(a)u_t = u_t a, \qquad a \in M, t \geq 0.$$

Notice that in this case the projective cocycle p is related to the semigroup u by $p_t = u_t u_t^*$, $t \geq 0$.

We will not use Powers' terminology in this paper because we are concerned with dilation theory, and in dilation theory the term *compression* carries a somewhat broader meaning. Moreover, our need for projective cocycles has grown from considerations that are quite different from those of [12], and it will be more convenient for us to deal directly with the cocycles rather than with their associated semigroups.

Definition 2.3. *Let $p = \{p_t : t > 0\}$ and $q = \{p_t : t > 0\}$ be two projective cocycles. We write $p \leq q$ if $p_t \leq q_t$ for every $t > 0$.*

The following proposition gives a general procedure for constructing projective cocycles in arbitrary von Neumann algebras from families of projections having somewhat less structure.

Proposition 2.4. *Let α be an E_0-semigroup acting on a von Neumann algebra M and let $\{f_t : t > 0\}$ be a family of nonzero projections in M satisfying*

$$f_t \in \alpha_t(M)' \tag{2.4.1}$$

$$f_{s+t} \leq f_s\alpha_s(f_t), \qquad s, t > 0. \tag{2.4.2}$$

Fix $t > 0$ and consider the set of all finite partitions

$$\mathcal{P} = \{0 < t_1 < t_2 < \cdots < t_n = t\}$$

of the interval $[0, t]$ as an increasing directed set in the usual sense. For such a partition $\mathcal{P}$, define an operator $f_{\mathcal{P}}$ by

$$f_{\mathcal{P}} = f_{t_1}\alpha_{t_1}(f_{t_2-t_1})\alpha_{t_2}(f_{t_3-t_2})\dots\alpha_{t_{n-1}}(f_{t_n-t_{n-1}}). \tag{2.4.3}$$

$f_{\mathcal{P}}$ is a projection and $\mathcal{P}_1 \subseteq \mathcal{P}_2 \implies f_{\mathcal{P}_1} \leq f_{\mathcal{P}_2}$. Thus we can define a projection p_t by

$$p_t = \sup_{\mathcal{P}} f_{\mathcal{P}} = \lim_{\mathcal{P}} f_{\mathcal{P}}.$$

The family $p = \{p_t : t > 0\}$ is a projective cocycle, and is the smallest projective cocycle p such that $f_t \leq p_t$ for every $t > 0$.

proof. Let $s, t > 0$ and let a and b be operators in M such that a commutes with $\alpha_s(M)$ and b commutes with $\alpha_t(M)$. Then a commutes with $\alpha_s(b)$ and note that the product $a\alpha_s(b)$ commutes with $\alpha_{s+t}(M)$. Indeed, for arbitrary $c \in M$ we have

$$a\alpha_s(b)\alpha_{s+t}(c) = a\alpha_s(b\alpha_t(c)) = \alpha_s(\alpha_t(c)b)a = \alpha_{s+t}(c)\alpha_s(b)a = \alpha_{s+t}(c)a\alpha_s(b).$$

Now fix $t > 0$. It follows from the preceding remarks that the operator $f_{\mathcal{P}}$ of (2.4.3) belongs to $M \cap \alpha_t(M)'$; moreover, the n factors of $f_{\mathcal{P}}$ on the right side of (2.4.3) are mutually commuting projections. Thus $f_{\mathcal{P}}$ is a projection in $M \cap \alpha_t(M)'$.

To show that $f_{\mathcal{P}}$ increases with $\mathcal{P}$ it is enough to show that if a given partition $\mathcal{P} = \{0 < t_1 < \cdots < t_n = t\}$ is refined by adjoining to it a single point τ, then $f_{\mathcal{P}}$ increases. In turn, that reduces to the following assertion. For $k = 1, 2, \dots, n$ and $t_{k-1} < \tau < t_k$ (where t_0 is taken as 0),

$$f_{t_k-t_{k-1}} \leq f_{\tau-t_{k-1}}\alpha_{\tau-t_{k-1}}(f_{t_k-\tau}).$$

The latter is immediate from the hypothesis (2.4.2).

Thus the net $f_{\mathcal{P}}$ increases with $\mathcal{P}$ and we can define a projection $p_t \in M \cap \alpha_t(M)'$ as asserted. The cocycle property (2.1.2) follows immediately from the definition of the family $\{p_t : t > 0\}$.

We obviously have $f_t \leq p_t$ for every $t > 0$. Finally, suppose $q = \{q_t : t > 0\}$ is another projective cocycle satisfying $f_t \leq q_t$ for every $t > 0$. Fix $t > 0$ and let

$\mathcal{P} = \{0 < t_1 < \cdots < t_n = t\}$ be a partition of the interval $[0, t]$. Then for every $k = 1, 2, \dots, n$ we have $f_{t_k - t_{k-1}} \le q_{t_k - t_{k-1}}$, and hence $f_\mathcal{P} \le q_\mathcal{P}$. On the other hand, the cocycle property of q implies that $q_\mathcal{P} = q_t$. Hence $f_\mathcal{P} \le q_t$ and we deduce the desired inequality

$$p_t = \sup_\mathcal{P} f_\mathcal{P} \le q_t.$$

□

The following result will be important for section 3. It implies that certain projections in M naturally give rise to projective cocycles.

Corollary 2.5. *Let e be a nonzero projection in M satisfying $\alpha_t(e) \ge e$ for every $t \ge 0$. For each $t > 0$ let f_t be the smallest projection in $M \cap \alpha_t(M)'$ which dominates e, i.e.,*

$$f_t = [\alpha_t(a)e\xi : a \in M, \xi \in H].$$

Then $f_{s+t} \le f_s \alpha_s(f_t)$ for every $s, t > 0$. The projective cocycle $p = \{p_t : t > 0\}$ of Proposition 2.4 is the smallest projective cocycle satisfying $p_t \ge e$ for every $t > 0$.

proof. It is obvious that f_t commutes with $\alpha_t(M)$, and the double commutant theorem implies that $f_t \in M$. Hence $f_t \in M \cap \alpha_t(M)'$.

To see that $f_{s+t} \le f_s \alpha_s(f_t)$, fix $s, t > 0$. By the argument at the beginning of the proof of proposition 2.4, $f_s \alpha_s(f_t)$ is a projection in $M \cap \alpha_{s+t}(M)'$. We claim that $e \le f_s \alpha_s(f_t)$. Indeed, $e \le f_s$ follows from the definition of f_s, and since $e \le f_t$ implies $\alpha_s(e) \le \alpha_s(f_t)$ we have $e \le \alpha_s(e) \le \alpha_s(f_t)$. Hence $e \le f_s \alpha_s(f_t)$. Since f_{s+t} is the smallest projection in $M \cap \alpha_{s+t}(M)'$ which dominates e we have the asserted inequality $f_{s+t} \le f_s \alpha_s(f_t)$ □

3. Minimality.

Throughout the section, α will denote an E_0-semigroup acting on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ and p will denote a projection in M satisfying

$$\alpha_t(p) \ge p, \qquad t \ge 0. \tag{3.1}$$

M_0 will denote the hereditary subalgebra pMp. We will be concerned with the (perhaps nonunital) von Neumann algebra M_+ generated by M_0 and its translates under α,

$$M_+ = \overline{\operatorname{span}}\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\dots\alpha_{t_n}(a_n) : a_k \in M_0, t_k \ge 0, n = 1, 2, \dots\},$$

the bar denoting closure in the weak operator topology. It is obvious that $\alpha_t(M_+) \subseteq M_+$ for every $t \ge 0$, and the unit of M_+ is the projection

$$p_\infty = \lim_{t\to\infty} \alpha_t(p). \tag{3.3}$$

There is a smaller projection that is of greater importance, namely

$$p_+ = [M_+ pH] = [\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\dots\alpha_{t_n}(a_n)\xi : a_k \in M_0, t_k \ge 0, \xi \in pH]. \tag{3.4}$$

Remarks. p_+ is the unit of the two-sided ideal $\overline{\operatorname{span}} M_+ p M_+$ of M_+ generated by p. Thus, p_+ belongs to the center of M_+, and in fact is the smallest central projection c in M_+ satisfying $p \le c$.

We will eventually show that p_+ is an increasing projection which defines a multiplicative compression of α. Neither of these assertions is apparent from (3.4). We deduce these properties from the following result which gives a "formula" for

Theorem 3.5. *Let p be a projection in M which satisfies (3.1), let p_∞ and p_+ be defined as in (3.3) and (3.4) respectively. Let $q = \{q_t : t > 0\}$ be the smallest projective cocycle satisfying $q_t \geq p$ for every $t \geq 0$ as in Corollary 2.5, and set*

$$q_\infty = \lim_{t\to\infty} q_t.$$

Then p_∞ belongs to the tail von Neumann algebra $M_\infty = \cap_{t\geq 0}\alpha_t(M)$, q_∞ belongs to its relative commutant in M, and we have a factorization

$$p_+ = p_\infty q_\infty.$$

Remarks. Recall that since q is a projective cocycle, q_t must be a decreasing function of t and hence the strong limit $q_\infty = \lim_{t\to\infty} q_t$ exists.

proof. Notice first that since $\alpha_t(p) \in \alpha_t(M)$ and since the von Neumann algebras $\alpha_t(M)$ decrease as t increases, it follows that $p_\infty = \lim_{t\to\infty} p_t \in M_\infty$. Since q_t belongs to $M \cap \alpha_t(M)' \subseteq M \cap M_\infty'$ for all t we see that $q_\infty = \lim_{t\to\infty} q_t \in M \cap M_\infty'$. In particular, the projections p_∞ and q_∞ must commute.

We show first that $p_+ \leq p_\infty q_\infty$. Since $p_+ \leq p_\infty$ is obvious, it suffices to show that $p_+H \leq q_\infty H$. Considering the definition of p_+ and the fact that $pH \leq q_\infty H$, it suffices to show that the subspace $q_\infty H$ is invariant under any operator in any one of the von Neumann algebras $\alpha_t(M_0)$, $t > 0$, i.e., that q_∞ commutes with $\cup_{t>0}\alpha_t(M_0)$. For that, fix $t > 0$. If we pass s to ∞ in the cocycle formula

$$q_t\alpha_t(q_s) = q_{t+s}$$

and use normality of α_t we obtain

$$q_t\alpha_t(q_\infty) = q_\infty. \tag{3.6}$$

It follows that for any $a \in M_0$ we have

$$\alpha_t(a)q_\infty = \alpha_t(a)q_t\alpha_t(q_\infty) = q_t\alpha_t(a)\alpha_t(q_\infty) = q_t\alpha_t(aq_\infty), \tag{3.7}$$

where we have used $q_t \in \alpha_t(M)'$. Now since $p \leq q_\infty$ and since $a = pap$ we have $aq_\infty = a = q_\infty a$. Thus we can replace the right side of (3.7) with

$$q_t\alpha_t(q_\infty a) = q_t\alpha_t(q_\infty)\alpha_t(a) = q_\infty\alpha_t(a).$$

Thus $\alpha_t(a)$ commutes with q_∞ as required.

It remains to show that $p_\infty q_\infty \leq p_+$. For that, it suffices to show that for every $t > 0$ we have

$$q_t\alpha_t(p_+) \leq p_+. \tag{3.8}$$

Indeed, assuming that (3.8) has been established we deduce $q_t\alpha_t(p) \leq p_+$ for every t (because $p \leq p_+$); noting that $q_t \downarrow q_\infty$ and $\alpha_t(p) \uparrow p_\infty$ as t increases to $+\infty$, we may take the strong limit on t in the previous formula to obtain the desired inequality $q_\infty p_\infty \leq p_+$.

In order to prove (3.8), we require

Lemma 3.9. *For each $t > 0$ let f_t be the projection onto the subspace*

$$[\alpha_t(a)p\xi : a \in M, \xi \in H].$$

Then $f_t\alpha_t(p_+) \leq p_+$.

proof. We have already pointed out in the remarks following (3.4) that p_+ is the unit of the ideal $\overline{\text{span}} M_+pM_+$ in M_+. Thus it suffices to show that

$$f_t\alpha_t(M_+pM_+)H \subseteq p_+H.$$

Since f_t commutes with $\alpha_t(M)$ the left side is contained in

$$\alpha_t(M_+pM_+)f_tH \subseteq [\alpha_t(M_+pM_+)\alpha_t(M)pH] \subseteq [\alpha_t(M_+pM_+M)pH].$$

Noting that $p = \alpha_t(p)p$ the latter is

$$\begin{aligned} [\alpha_t(M_+pM_+Mp)pH] &\subseteq [\alpha_t(M_+pMp)pH] \subseteq [\alpha_t(M_+)pH] \\ &\subseteq [M_+pH] = p_+H, \end{aligned}$$

as asserted □

For every $t > 0$ we define a normal linear mapping $\beta_t : M \to M$ as follows,

$$\beta_t(a) = f_t\alpha_t(a).$$

Since f_t commutes with $\alpha_t(M)$, β_t is an endomorphism of the $*$-algebra structure of M for which $\beta_t(\mathbf{1}) = f_t$, but it is not a semigroup because $\{f_t : t > 0\}$ does not satisfy the cocycle condition (1.1.2). However, because of Lemma 3.9 we have

$$\beta_t(p_+) \leq p_+, \qquad t > 0.$$

Now fix $t > 0$ and let $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of the interval $[0, t]$. By iterating the preceding formula we find that

$$\beta_{t_1}\beta_{t_2-t_1}\beta_{t_3-t_2}\dots\beta_{t_n-t_{n-1}}(p_+) \leq p_+. \tag{3.10}$$

In the notation of Proposition 1.4, the left side of (3.10) is

$$f_{t_1}\alpha_{t_1}(f_{t_2-t_1})\alpha_{t_2}(f_{t_3-t_2})\dots\alpha_{t_{n-1}}(f_{t_n-t_{n-1}})\alpha_t(p_+) = f_{\mathcal{P}}\alpha_t(p_+).$$

Using 1.4 and 1.5, we make take the limit on $\mathcal{P}$ in (3.10) obtain the required inequality (3.8)

$$q_t\alpha_t(p_+) = \lim_{\mathcal{P}} f_{\mathcal{P}}\alpha_t(p_+) \leq p_+,$$

completing the proof of Theorem 3.5 □

We can now deduce the following result, which paraphrases Theorem A from section 1.

Theorem A. *Let p be an increasing projection for α and let $p_+ \geq p$ be the projection defined by (3.4). p_+ is an increasing projection with the property that the compression of α to p_+Mp_+ is multiplicative.*

If r is another increasing projection in M such that $r \geq p$ and the compression of α to rMr is multiplicative, then $r \geq p_+$.

proof. Since $p_\infty = \lim_{t\to\infty} \alpha_t(p)$ is clearly fixed under the action of α_t and since (3.6) implies that $\alpha_t(q_\infty) \geq q_\infty$, we find that $\alpha_t(p_+) = \alpha_t(p_\infty)\alpha_t(q_\infty) \geq p_\infty q_\infty = p_+$.

To show that the compression of α to p_+Mp_+ is multiplicative, it suffices to show that p_+ commutes with $\alpha_t(p_+Mp_+)$ for every $t > 0$ (Propostion 1.7). For that, it is enough to show that for every $a = a^* \in p_+Mp_+$ we have

$$p_+\alpha_t(a) = q_t\alpha_t(a). \tag{3.11}$$

Indeed, by taking adjoints in (3.11) we find that $\alpha_t(a)q_t = \alpha_t(a)p_+$, and since q is a projective cocycle q_t must commute with $\alpha_t(M)$. Thus

$$p_+\alpha_t(a) = q_t\alpha_t(a) = \alpha_t(a)q_t = \alpha_t(a)p_+,$$

and thus $p_+ \in \alpha_t(p_+Mp_+)'$.

To prove (3.11), we write $p_+ = p_\infty q_\infty = q_\infty p_\infty$ and use $\alpha_t(p_\infty) = p_\infty$ to obtain

$$p_+\alpha_t(a) = q_\infty p_\infty \alpha(t(a) = q_\infty\alpha_t(p_\infty a) = q_\infty\alpha_t(a), \tag{3.12}$$

because $p_\infty a = a$ for every operator a in $p_+Mp_+ \subseteq p_\infty M p_\infty$. Using (3.6) on the last term of (3.12) we have

$$q_\infty\alpha_t(a) = q_t\alpha_t(p_\infty)\alpha_t(a) = q_t\alpha_t(q_\infty a) = q_t\alpha_t(a),$$

since $q_\infty a = a$ for every $a \in p_+Mp_+ \subseteq q_\infty M q_\infty$. Formula (3.11) follows.

Finally, let $r \geq p$ be another increasing projection with the property that the compression of α to rMr is multiplicative. We have to show that $p_+H \subseteq rH$. Because of formula (3.4) for p_+H, together with the fact that $pH \subseteq rH$, it is enough to show that rH is invariant under any operator of the form $\alpha_t(a)$ with $a \in pMp$ and $t > 0$. But for each $t > 0$, Proposition 1.7 implies that r commutes with the set of operators $\alpha_t(rMr)$, and therefore since $p \leq r$ we have

$$\alpha_t(pMp)rH \subseteq \alpha_t(rMr)rH \subseteq rH,$$

as required $\square$

As another consequence of Theorem 3.5 we have the following characterization of minimality in terms of projective cocycles.

Corollary 3.13. *Let p be an increasing projection and let ϕ be the compression of α to pMp. Then α is minimal over ϕ iff $\lim_{t\to\infty}\alpha_t(p) = \mathbf{1}$ and the only projective cocycle $q = \{q_t : t > 0\}$ satisfying $q_t \geq p$ for every $t > 0$ is the trivial cocycle $q_t = \mathbf{1}$.*

proof. The minimality assertion is that $p_+ = \mathbf{1}$ and from Theorem 3.5 we have $p_+ = p_\infty q_\infty$. Thus α is minimal iff $p_\infty = q_\infty = \mathbf{1}$ $\square$

Proposition 3.14. *Let M_+ be the following von Neumann subalgebra of M*

$$M_+ = \overline{span}\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\dots\alpha_{t_n}(a_n) : a_1,\dots,a_n \in pMp, t_1,\dots,t_n \geq 0, n \geq 1\},$$

and let p_+ be the projection of (3.4). Then p_+ is the smallest projection in the center of M_+ which dominates p, and we have $p_+Mp_+ = M_+p_+$.

proof. While the first assertion is very elementary, we include a proof for completeness. Clearly $p_+ \in M_+$, and since $p_+H = [M_+pH]$ is invariant under M_+ we have $p_+ \in M_+'$. Hence p_+ belongs to the center of M_+ and $p \leq p_+$. If c is another central projection in M_+ for which $c \geq p$, then cH clearly contains $[M_+pH] = p_+H$ and hence $c \geq p_+$.

Let R be the weakly closed subspace of M generated by the set of operators

$$\{apb : a \in M_+, b \in M\}.$$

R is a right ideal in M, and the range projection of R is

$$[RH] = [apH : a \in M_+] = p_+H.$$

Hence $R = p_+M$. Thus p_+Mp_+ is spanned by RR^*, i.e.,

$$p_+Mp_+ = \overline{span}(M_+ \cdot pMp \cdot M_+) = \overline{span}(M_+pM_+).$$

The right side of the preceding formula is the two-sided ideal in M_+ generated by p which, by the preceding paragraph, is M_+p_+ $\square$

Theorem B. *Let α be an E_0-semigroup acting on a factor $M \subseteq \mathcal{B}(H)$, H being a separable Hilbert space. Let $p \in M$ be an increasing projection and let ϕ be the compression of α to pMp. The following are equivalent.*

(1) *α is minimal over ϕ.*
(2) *H is spanned by*

$$\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\dots\alpha_{t_n}(a_n)\xi : a_1,\dots,a_n \in pMp, t_k \geq 0, n \geq 1, \xi \in pH\}.$$

(3) *M is generated as a von Neumann algebra by the set of operators*

$$\{\alpha_t(a) : a \in pMp, t \geq 0\}.$$

proof of (1) $\Longrightarrow$ (3). If α is minimal over ϕ then $p_+ = \mathbf{1}$, and thus by Proposition 3.14 we find that $M = p_+Mp_+ = M_+p_+ = M_+$, hence (3).

proof of (3) $\Longrightarrow$ (2). Since $M = M_+$ we have $[M_+pH] = [MpH]$. The projection on the subspace on the right is the central carrier of p, which must be $\mathbf{1}$ because M is a factor. Therefore $p_+H = H$, as asserted in (2).

proof of (2) $\Longrightarrow$ (1). The hypothesis is that $p_+ = \mathbf{1}$ which, by Theorem A, implies that α is minimal over ϕ $\square$

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