E_0-SEMIGROUPS IN QUANTUM FIELD THEORY

William Arveson

Department of Mathematics
University of California
Berkeley CA 94720, USA

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Abstract. A lecture presented at the von Neumann symposium on Quantization and Nonlinear Wave Equations held at MIT in June, 1994. We describe the role of semigroups of endomorphisms of von Neumann algebras in algebraic formulations of quantum field theory, and present a summary of recent developments in the theory of E_0-semigroups.

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1. Relation to symmetry groups of algebras of local observables.

In the algebraic formulation of quantum field theory, one works with a C^*-algebra $\mathcal{A}$ that is acted upon by a group of automorphisms representing the symmetries of spacetime (for example the Poincaré group), and which is also given a local structure (subalgebras of local observables). This local structure normally consists of associating a (von Neumann) subalgebra of $\mathcal{A}$ to every bounded region of spacetime in a coherent way, so that inclusions match up correctly, so that these subalgebras transform covariantly under the group action, so that the topologies fit together consistently, and which may satisfy additional conditions arising from the underlying physics. This C^*-algebra is not necessarily presented in a representation on a Hilbert space. However, given any state of $\mathcal{A}$ which restricts to a normal state on the local subalgebras, one can perform the usual GNS construction and obtain a representation of this algebra on a Hilbert space H.

Typically, the group of automorphisms (or large subgroups of it) leave the state invariant, and therefore one can naturally construct a unitary representation of the group on H which implements the action. Moreover, once one has a concrete representation of $\mathcal{A}$, one can associate von Neumann algebras with every (not necessarily bounded) open subset of spacetime by taking weak closures of appropriate unions.

For example, there is a von Neumann algebra $\mathcal{M}(C) \subseteq \mathcal{B}(H)$ associated with the forward light cone

$$C = \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : (x_1^2 + x_2^2 + x_3^2)^{1/2} < t\ \}.$$

If v is any vector of $\mathbb{R} \times \mathbb{R}^3$ which lies inside C then the one parameter group $\lambda \in \mathbb{R} \mapsto \lambda v$ defines a physical flow of time. This flow is represented by a one-parameter unitary group $\{U_\lambda : \lambda \in \mathbb{R}\ \}$ acting on H. Since $C + \lambda v \subseteq C$ for $\lambda \geq 0$, we may conclude from the covariance properties of the given structure that for *nonnegative* λ we have

$$U_\lambda \mathcal{M}(C) U_\lambda^* \subseteq \mathcal{M}(C). \tag{1.1}$$

Thus (1.1) defines a semigroup of isometric unit-preserving $*$-endomorphisms which acts on the von Neumann algebra $\mathcal{M}(C)$. In the simplest case where $\mathcal{M}(C)$ is a factor of type I_∞, we have a prototypical example of an E_0-semigroup $\{\alpha_\lambda : \lambda \geq 0\ \}$

$$\alpha_\lambda(A) = U_\lambda A U_\lambda^*, \qquad A \in \mathcal{M}(C), \quad \lambda \geq 0.$$

We will take up this situation in a more general setting. By an E_0-semigroup we mean a semigroup $\{\alpha_t : t \geq 0\ \}$ of self-adjoint normal endomorphisms of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H which preserves the identity ($\alpha_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$) and is such that

$$t \mapsto < \alpha_t(A)\xi, \eta >$$

is continuous for every $\xi, \eta \in H$ and every $A \in \mathcal{B}(H)$. We emphasize that for the results discussed in the sequel, it is essential that the Hilbert space H should be separable. Naturally, one may speak of E_0-semigroups that act on type I factors $\mathcal{M}$, and in this less concrete situation we require that the predual of $\mathcal{M}$ be separable. Two E_0-semigroups α (acting on $\mathcal{M}$) and β (acting on $\mathcal{N}$) are said to be *conjugate* if there is a $*$-isomorphism $\theta : \mathcal{M} \to \mathcal{N}$ such that $\theta \circ \alpha_t = \beta_t \circ \theta$ for every $t \geq 0$. One may also speak of E_0-semigroups that act on more general factors, but we shall not do so here.

The preceding discussion shows that one may obtain E_0-semigroups by starting with a strongly continuous one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ acting on a Hilbert space H, finding somehow a type I subfactor $\mathcal{M}_+ \subseteq \mathcal{B}(H)$ which is invariant in the sense that

$$U_t \mathcal{M}_+ U_t^* \subseteq \mathcal{M}_+$$

for nonnegative t and then defining an E_0-semigroup α on $\mathcal{M}_+$ by setting

$$\alpha_t(A) = U_t A U_t^*, \qquad t \geq 0, \quad A \in \mathcal{M}_+.$$

Notice that in this case there is a "complementary" E_0-semigroup. Indeed, the commutant $\mathcal{M}_- = \mathcal{M}_+'$ of $\mathcal{M}_+$ is also a type I factor, and is invariant under the automorphisms $A \mapsto U_t A U_t^*$ for *negative* t. Thus we can define a second E_0-semigroup β acting on $\mathcal{M}_-$ by way of

$$\beta_t(A) = U_t^* A U_t, \qquad t \geq 0, \quad A \in \mathcal{M}_-.$$

These remarks show that the E_0-semigroups that arise in this way from automorphism groups of a larger type I factor always occur in pairs. Several natural questions occur at this point. For example, does *every* abstract E_0-semigroup α which acts on a type I factor arise in this way from a one parameter group of automorphisms of a larger type I factor? If it does, then what is the relation between the "positive" and "negative" semigroups α and β? We will answer these and other questions presently. Let us first recall some known results pertaining to an analogous but much simpler situation.

Let $U = \{U_t : t \geq 0\}$ be a strongly continuous one parameter semigroup of isometries acting on a Hilbert space H. Then H can be decomposed into a direct sum $H_1 \oplus H_2$ giving rise to a corresponding decomposition

$$U = V \oplus W \tag{1.2}$$

of U into a direct sum of two semigroups of isometries with the property that V is *pure* in that

$$\bigcap_{t \geq 0} V_t H_1 = 0$$

and such that W_t is unitary for every $t \geq 0$. This decomposition is unique. Moreover, the pure summand V is unitarily equivalent to a direct sum of a countable number n of copies of the *shift* semigroup S, which acts on the Hilbert space $L^2(0, \infty)$ by way of

$$S_t f(x) = \begin{cases} f(x-t), & \text{for } x > t \\ 0, & \text{for } 0 < x \leq t. \end{cases}$$

The number n is also uniquely determined. Taken together, these results are often called the Wold decomposition. The summand H_2 is defined by

$$H_2 = \bigcap_{t \geq 0} U_t H,$$

and of course H_1 is the orthocomplement of H_2.

One of my first thoughts about E_0-semigroups was to speculate that there should be an effective decomposition resembling the Wold decomposition, and that this might lead to some kind of classification up to conjugacy. It quickly became clear that this idea was too naive. Since a discussion of this issue serves to make a significant point about E_0-semigroups, we offer the following comments.

Suppose that one is given an E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ which acts on a type I factor $\mathcal{M}$. We indicate the extent to which a decomposition resembling the Wold decomposition is valid for α. There are two natural von Neumann algebras associated with the action of α:

$$\mathcal{M}_\infty = \bigcap_{t \geq 0} \alpha_t(\mathcal{M}), \qquad \text{and}$$
$$\mathcal{M}^\alpha = \{A \in \mathcal{M} : \alpha_t(A) = A, t \geq 0\}.$$

$\mathcal{M}_\infty$ is the "tail" von Neumann algebra, and $\mathcal{M}^\alpha$ is the fixed algebra. We have $\mathcal{M}^\alpha \subseteq \mathcal{M}_\infty \subseteq \mathcal{M}$, and the action of α_t on $\mathcal{M}_\infty$ defines an automorphism of $\mathcal{M}_\infty$ for every $t \geq 0$. Thus the restriction of α to $\mathcal{M}_\infty$ defines a one-parameter *group* of automorphisms of $\mathcal{M}_\infty$. This W^*-dynamical system plays the role of the summand W in the Wold decomposition (1.2), and it is clearly a conjugacy invariant of the original E_0-semigroup α.

Let $\mathcal{N} = \mathcal{M}_\infty'$ denote the (relative) commutant of $\mathcal{M}_\infty$. Notice that since $\mathcal{M}$ is a factor of type I, we may treat relative commutants (in $\mathcal{M}$) as if they were true commutants. Since $\alpha_t(\mathcal{M}_\infty) = \mathcal{M}_\infty$ for every $t \geq 0$, it follows in a straightforward way that $\alpha_t(\mathcal{N}) \subseteq \mathcal{N}$, $t \geq 0$. Moreover, it is not hard to show that the restriction of α to $\mathcal{N}$ has trivial "tail" in the sense that

$$\bigcap_{t \geq 0} \alpha_t(\mathcal{N}) = \text{center of } \mathcal{N} = \text{center of } \mathcal{M}_\infty.$$

In the important case where $\mathcal{M}_\infty$ is a factor, then we have a decomposition resembling (1.2) to some extent, in that

$$\mathcal{M} = \mathcal{N} \vee \mathcal{M}_\infty \tag{1.3}$$

is generated by a pair of mutually commuting factors which are commutants of each other, that α determines an automorphism group on $\mathcal{M}_\infty$, that α leaves $\mathcal{N}$ invariant and in fact defines a "pure" semigroup on $\mathcal{N}$ in the sense that

$$\bigcap_{t \geq 0} \alpha_t(\mathcal{N}) = \mathbb{C}\mathbf{1}. \tag{1.4}$$

If $\mathcal{M}_\infty$ happens to be a factor of type I then (1.3) becomes a simple tensor product

$$\mathcal{M} = \mathcal{N} \otimes \mathcal{M}_\infty, \tag{1.5}$$

and if we define E_0-semigroups β on $\mathcal{N}$ and γ on $\mathcal{M}_\infty$ by restricting α in the obvious way then we have a decomposition

$$\alpha = \beta \otimes \gamma. \tag{1.6}$$

Formula (1.6) appears quite analogous to the Wold decomposition (1.2).

Unfortunately, $\mathcal{M}_\infty$ need *not* be a type I von Neumann algebra. Moreover, even if it were a type I factor, we do not know how to classify E_0-semigroups that satisfy (1.4). And if $\mathcal{M}_\infty$ is a factor not of type I, then almost nothing is known about semigroups satisfying (1.4). Finally, it is known that there exist E_0-semigroups whose fixed algebra $\mathcal{M}^\alpha$ is a factor of type II_∞ or of type III (we will have more to say about this in sections 7 and 8).

These remarks lead one to the conclusion that E_0-semigroups are much too complex to hope for a useful classification up to *conjugacy*.

2. Cocycle perturbations.

Let α be an E_0-semigroup acting on $\mathcal{B}(H)$. A *cocycle* for α is a *strongly continuous* family of unitary operators $U = \{U_t : t \geq 0\}$ satisfying

$$U_{s+t} = U_s\alpha_s(U_t), \qquad s, t \geq 0. \tag{2.1}$$

Notice that (2.1) implies that $U_0 = \mathbf{1}$. The condition also implies that the family of endomorphisms $\beta = \{\beta_t :\geq 0\}$ defined by

$$\beta_t(A) = U_t\alpha_t(A)U_t^*, \qquad t \geq 0 \tag{2.2}$$

is another E_0-semigroup acting on $\mathcal{B}(H)$. β is called a *cocycle perturbation* of α. Two E_0-semigroups are said to be *cocycle conjugate* if one of them is conjugate to a cocycle perturbation of the other. This is an important relation for the theory of E_0-semigroups, and we want to discuss the notion of cocycle perturbations in more detail.

Suppose that α and β are two E_0-semigroups which act on the same $\mathcal{B}(H)$, and that β is a cocycle perturbation of α as in (2.2):

$$\beta_t(A) = U_t\alpha_t(A)U_t^*, \qquad A \in \mathcal{B}(H), \quad t \geq 0.$$

Notice that the unitary operators $V_t = U_t^*$, $t \geq 0$ define a cocycle for β and of course we have

$$\alpha_t(A) = V_t\beta_t(A)V_t^*.$$

Thus, this notion of cocycle perturbation defines an equivalence relation on the set of all E_0-semigroups that act on a fixed $\mathcal{B}(H)$.

Note too that this relation is analogous to Connes' notion of exterior equivalence for one-parameter automorphism groups of a von Neumann algebra $\mathcal{M}$ [13]. In particular, Connes discovered that if one is given a pair of faithful normal weights of $\mathcal{M}$, then the two associated modular automorphism groups are exterior equivalent, and this led to the establishment of an elegant classification theory for type III factors. Cocycle perturbations are of fundamental importance for the classification of E_0-semigroups as well, but for reasons that involve new phenomena that are unique to the theory of E_0-semigroups (see Theorem B below).

Remark 2.3. The cohomology of cocycle perturbations. Suppose first that α and β are *conjugate* E_0-semigroups which act on the same $\mathcal{B}(H)$, say $\theta \circ \beta_t = \alpha_t \circ \theta$ where θ is a $*$-automorphism of $\mathcal{B}(H)$. We may find a unitary operator W that implements θ in the sense that $\theta(A) = W^*AW$, $A \in \mathcal{B}(H)$. It follows that

$$\beta_t(A) = U_t\alpha_t(A)U_t^*, \qquad t \geq 0 \quad A \in \mathcal{B}(H),$$

where U is the α-cocycle defined by $U_t = W\alpha_t(W)^*$. An α-cocycle of this form is called *exact*. Conversely, if β is a perturbation of α by an exact cocycle, then α and β are conjugate. This remark explains why the problem of classifying E_0-semigroups to conjugacy is difficult: it is the same as the problem of computing a very subtle noncommutative cohomology group.

Remark 2.4. Perturbations of the infinitesimal generator. We want to point out a useful interpretation of cocycle conjugacy in terms of infinitesimal generators. Suppose that α acts on $\mathcal{B}(H)$, and let δ be the generator of α. For an operator A in the domain of δ, $\delta(A)$ is defined as the limit in the strong operator topology

$$\delta(A) = \lim_{t\to 0+} t^{-1}(\alpha_t(A) - A).$$

The domain of δ is a unital $*$-subalgebra $\mathcal{D}$ of $\mathcal{B}(H)$ and δ is a self-adjoint derivation of $\mathcal{D}$ into $\mathcal{B}(H)$. Let B be any bounded self adjoint operator in $\mathcal{B}(H)$ and put

$$\delta'(A) = \delta(A) + i(BA - AB).$$

δ' is another unbounded self-adjoint derivation having the same domain $\mathcal{D}$, and it can be shown that it is the generator of a second E_0-semigroup β which acts on $\mathcal{B}(H)$. In fact, β is a cocycle perturbation of α and the cocycle U relating β to α as in (2.2) can be defined as the global solution of the linear differential equation

$$\frac{d}{dt}U(t) = iU(t)\alpha_t(B), \qquad t \geq 0,$$

with the initial condition $U(0) = \mathbf{1}$. It is easy to see from the properties of this differential equation and the fact that B is bounded that the cocycle U is *norm* continuous:

$$\lim_{t\to t_0} \|U(t) - U(t_0)\| = 0,$$

for every $t_0 \geq 0$. Now in general, cocycles need not be norm continuous, and the generators of cocycle perturbations of α cannot be obtained by perturbing the generator of α by bounded derivations. Nevertheless, it is useful to think of the cocycle perturbations of an E_0-semigroup as having been obtained by perturbing its generator by an "unbounded" derivation. While this interpretation is merely a heuristic conceptual device, it can sometimes be made precise. In any event, one should consider the definition of cocycle perturbation as a precise formulation of this idea.

3. Numerical index.

The preceding discussion shows that one should look for cocycle conjugacy invariants. We now describe a numerical index invariant which is appropriately thought of as a quantized form of the Fredholm index of certain differential operators (or of the operator semigroups that they generate). It is defined as follows.

Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(H)$. A *unit* for α is a strongly continuous semigroup of bounded operators $\{U_t : t \geq 0\}$ on H which intertwines α in the sense that

$$\alpha_t(A)U_t = U_tA, \qquad t \geq 0, \quad A \in \mathcal{B}(H),$$

and is not the zero semigroup $U_t = 0$. $\mathcal{U}_\alpha$ will denote the set of all units of α. Notice that if U and V belong to $\mathcal{U}_\alpha$ then for each $t \geq 0$ $V_t^* U_t$ is a bounded operator which, by 3.1, commutes with every bounded operator on H, and hence must be a scalar multiple of the identity

$$V_t^* U_t = f(t)\mathbf{1}, \qquad t \geq 0.$$

f is a continuous complex-valued funciton satisfying $f(0) = 1$, and one easily verifies that $f(s + t) = f(s)f(t)$ for all nonnegative s, t. Hence there is a unique complex number $c(U, V)$ satisfying

$$f(t) = e^{c(U,V)t},$$

for every $t \geq 0$. The bivariate function

$$c : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}$$

is called the *covariance function* of α. The covariance function is easily seen to be conditionally positive definite. Thus one may use the pair $(\mathcal{U}_\alpha, c)$ in a natural way to construct a Hilbert space H_α. It can be shown that H_α is *separable*. We define

$$d_*(\alpha) = \dim H_\alpha.$$

Thus the possible values of $d_*(\alpha)$ are $0, 1, 2, \dots, \aleph_0$. Significantly, there are E_0-semigroups α for which $\mathcal{U}_\alpha = \emptyset$ [18], and in this case it is convenient for the arithmetic of the index to define $d_*(\alpha) = 2^{\aleph_0}$. See [2] for more detail.

The basic property of this index is that it behaves well under the formation of tensor products [6]. If α and β are two E_0-semigroups acting, respectively, on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ then it is easy to see that there is a unique E_0-semigroup $\alpha \otimes \beta$, acting on $\mathcal{B}(H \otimes K)$, which satisfies

$$(\alpha \otimes \beta)_t : A \otimes B \mapsto \alpha_t(A) \otimes \beta_t(B)$$

for every $t \geq 0, A \in \mathcal{B}(H), B \in \mathcal{B}(K)$.

Theorem A.

$$d_*(\alpha \otimes \beta) = d_*(\alpha) + d_*(\beta).$$

Notice that $d_*(\alpha)$ is invariant under cocycle conjugacy, or what is in substance the same, that $d_*(\alpha)$ is stable under cocycle perturbations of α. That is easily seen as follows. Let $W = \{W_t : t \geq 0\}$ be any cocycle for α and consider the associated perturbation β

$$\beta_t(A) = W_t \alpha_t(A) W_t^*, \qquad t \geq 0, \quad A \in \mathcal{B}(H).$$

One may check that if $U = \{U_t : t \geq 0\}$ is a unit for α and if we define

$$\tilde{U}_t = W_t U_t, \qquad t \geq 0,$$

then $\tilde{U}$ is a unit for β. Moreover, $U \mapsto \tilde{U}$ is a bijection of $\mathcal{U}_\alpha$ onto $\mathcal{U}_\beta$ which carries one covariance function into the other. Thus the Hilbert spaces H_α and H_β are isomorphic and it follows that $d_*(\alpha) = d_*(\beta)$.

This definition of numerical index is equivalent to the one given in [3], and differs significantly from Powers' earlier definition of numerical index [18]. In the latter the index was defined as the multiplicity of a certain representation of a C^*-algebra associated with the infinitesimal generator of α. Since the representation depended on making a particular choice of a unit, it was not clear that this index was unambiguously defined. Later, Powers and Robinson [22] gave another definition of index which was obviously well-defined, but which took values in an abstract set of equivalence classes rather than in the nonnegative integers. Recently, Powers and Price [23] have shown that $d_*(\alpha)$ actually agrees with Powers' "infinitesimal" definition in all cases. In particular, it is now clear that Powers' original definition of the numerical index was unambiguous.

4. Continuous tensor product systems.

We now want to emphasize a fundamental relationship between E_0-semigroups and continuous tensor products of Hilbert spaces. Indeed, one could argue that up to cocycle conjugacy, the theory of E_0-semigroups *is* the theory of continuous tensor products of Hilbert spaces.

Heuristically, a *product system* is a measurable family of Hilbert spaces $E = \{E_t : t > 0\}$ which behaves as if each E_t were a continuous tensor product

$$E_t = \bigotimes_{0<s<t} H_s, \qquad H_s = H$$

of copies of a single separable Hilbert space H. While this heuristic picture is often useful, one must be careful not to push it too far. Indeed, we will see that this picture is basically correct for the simpler examples of product systems, but that there are other examples with the remarkable property that the "germ" H fails to exist. Rather than reiterate the details of the precise definition here, we illustrate the essentials of the structure of product systems in the discrete case, where the positive real line is replaced with the discrete set $\mathbb{N} = \{1, 2, \dots\}$ of positive integers. Then we will indicate briefly how to change the axoims to pass from $\mathbb{N}$ to $\mathbb{R}^+$. Full details can be found in [2].

Let H be a separable Hilbert space. For every $n = 1, 2 \dots$ let $E(n)$ be the full tensor product of n copies of H:

$$E(n) = \underbrace{H \otimes H \otimes \cdots \otimes H}_{n \text{ times}}.$$

We may organize these spaces into a family of Hilbert spaces $p : E \to \mathbb{N}$ over $\mathbb{N}$ by setting

$$E = \{(t, \xi) : t \in \mathbb{N}, \xi \in E(t)\},$$

with projection $p(t, \xi) = t$. We introduce an associative multiplication on the structure E by making use of the tensor product

$$(s, \xi) \cdot (t, \eta) = (s + t, \xi \otimes \eta),$$

$\xi \in E(s)$, $\eta \in E(t)$. This multiplication is bilinear on fibers, and has the two additional properties

$$E(s + t) = \overline{\text{span}}\ E(s) \cdot E(t), \qquad s, t \in \mathbb{N} \tag{4.1}$$

$$< ux, vy > = < u, v >< x, y >, \quad \forall u, v \in E(s), \quad x, y \in E(t). \tag{4.2}$$

Notice that the Hilbert space associated with the sections of $p : E \to \mathbb{N}$ is the direct sum

$$\sum_{t \in \mathbb{N}} E(t) = \sum_{n=1}^{\infty} H^{\otimes n},$$

essentially the full Fock space over the one-particle space H.

A *unit* is a section $n \in \mathbb{N} \mapsto u_n \in E(n)$ satisfying

$$u_{m+n} = u_m u_n,$$

and which is not the zero section. The most general unit has the form

$$u_n = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n \text{ times}}$$

$n \geq 1$, where x is a nonzero element of the one-particle space H.

Fix $n = 1, 2 \dots$ and let u be a vector in E_n. u is called *decomposable* if for every $k = 1, 2, \dots, n-1$ there are vectors $v_k \in E(k)$, $w_k \in E(n-k)$ such that

$$u = v_k w_k.$$

Notice that the most general decomposable vector in $E(n)$ is an elementary tensor of the form

$$u = x_1 \otimes x_2 \otimes \cdots \otimes x_n,$$

where $x_k \in H$ for $k = 1, 2, \dots, n$.

A product system is a similar structure, except that it is associated with the space of positive reals rather than $\mathbb{N}$. More precisely, a *product system* is a family of separable Hilbert spaces over the semi-infinite interval $(0, \infty)$

$$p : E \to (0, \infty)$$

which admits an associative multiplication that is bilinear on fiber spaces and has properties analogous to properties corresponding to (4.1) and (4.2). Additionally, E is endowed with a *standard Borel structure* which is compatible in the natural way with the other structures of E, and with the further property that there should be a separable Hilbert space H such that

$$E \cong (0, \infty) \times H, \tag{4.3}$$

where $\cong$ denotes an isomorphism of measurable families of Hilbert spaces. Condition (4.3) is nontrivial, and is the property in this category that corresponds to local triviality of Hermitian vector bundles. There is a natural notion of isomorphism for the category of product systems, which we will not write down explicitly here (see [2]). We may also define *units* of E and *decomposable* vectors of the fiber spaces E_t in a way analogous to the above. For example, a unit is a measurable cross section

$$u : t \in (0, \infty) \mapsto u(t) \in E_t$$

satisfying $u(s + t) = u(s)u(t)$ for all $s, t > 0$, and which is not the zero section.

One might expect that it should be possible to write down a comprehensive list of (continuous) product systems as we have done above for their discrete analogues. In that case there is, up to isomorphism, exactly one "discrete" product system for every integer $d = 1, 2, \dots, \aleph_0$. d can be taken to be the dimension of the one-particle space E_1. In the continuous case, however, nothing like that is true. While there is a family of "natural" examples parameterized by the values $d = 1, 2, \dots, \aleph_0$, there are many others as well. The problem of classifying general product systems is an unsolved problem which, as we will see, is of central importance in the theory of E_0-semigroups.

Every E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ gives rise to a product system E_α in the following way. Suppose α acts on $\mathcal{B}(H)$. For every $t > 0$, let $E_\alpha(t)$ be the intertwining space

$$E_\alpha(t) = \{T \in \mathcal{B}(H) : \alpha_t(A)T = TA \quad \forall A \in \mathcal{B}(H)\}.$$

$E_\alpha(t)$ is obviously a complex vector space, and in fact it is a Hilbert space. Indeed, if $S, T \in E_\alpha(t)$ then because of the intertwining property it follows that T^*S commutes with every operator $A \in \mathcal{B}(H)$. Hence there is a unique complex number $< S, T >$ such that

$$T^*S = < S, T > \mathbf{1}.$$

$<, >$ is an inner product on $E_\alpha(t)$ which makes it into a Hilbert space. Thus we have a family of Hilbert spaces $p : E_\alpha \to (0, \infty)$ defined by

$$E_\alpha = \{(t, \xi) : t > 0, \quad \xi \in E_\alpha(t)\}$$

where $p(t, \xi) = t$. If we use operator multiplication to define multiplication in E_α by

$$(s, S) \cdot (t, T) = (s + t, ST),$$

then it is not hard to establish the properties (4.1) and (4.2). E_α inherits a natural standard Borel structure as a subspace of $(0, \infty) \times \mathcal{B}(H)$, where $\mathcal{B}(H)$ is endowed with the Borel structure generated by its weak* topology. Finally, it is a nontrivial fact that property (4.3) is valid as well [2]. Thus E_α is a product system. The importance of product systems in this context derives from the following result from [2].

Theorem B. *α and β are cocycle conjugate iff E_α and E_β are isomorphic product systems.*

In order to illustrate what lies behind Theorem B, let us consider the case in which α and β both act on $\mathcal{B}(H)$ and β is a cocycle perturbation of α:

$$\beta_t(A) = U_t \alpha_t(A) U_t^*, \qquad t \geq 0, \quad A \in \mathcal{B}(H),$$

where $U = \{U_t : t \geq 0\}$ is an α-cocycle. In this case it is quite easy to exhibit an isomorphism of product systems $\theta : E_\alpha \to E_\beta$. Indeed, if we fix $t > 0$ and choose $T \in E_\alpha(t)$, then one may verify directly that $U_t T$ belongs to $E_\beta(t)$. Thus we can define a unitary operator $\theta_t : E_\alpha(t) \to E_\beta(t)$ by $\theta_t(T) = U_t T$; θ is defined as the total map. θ is a (measurable) bijection which is unitary on fibers, hence it is an isomorphism of families of Hilbert spaces. The cocycle property implies that for $s, t > 0$ and $S \in E_\alpha(s), T \in E_\alpha(t)$ we have

$$\begin{aligned}\theta_s(S)\theta_t(T) &= U_s S U_t T = U_s \alpha_s(U_t) ST \\ &= U_{s+t} ST = \theta_{s+t}(ST).\end{aligned}$$

Thus θ preserves multiplication, and hence it is an isomorphism of product systems.

To prove the converse direction (still assuming that α and β act on the same $\mathcal{B}(H)$), one basically has to start with an isomorphism $\theta : E_\alpha \to E_\beta$ and show that θ is associated with a unitary α-cocycle U as above. This is technically more difficult, but the basic idea is similar (see [2], Theorem 3.18).

Theorem B implies that the problem of classifying E_0-semigroups up to cocycle conjugacy is equivalent to that of classifying *certain* product systems...namely, those product systems that can be associated with an E_0-semigroup as above. It is a basic result in our approach to E_0-semigroups that *every* product system arises in this way.

Theorem C. *For every product system E there is an E_0-semigroup α such that E is isomorphic to E_α.*

The proof of Theorem C is very indirect [5], and makes essential use of the spectral C^*-algebras discussed in sections 7 and 8.

We want to point out that there is a general notion of *dimension* of an abstract product system that generalizes the numerical index of E_0-semigroups. This dimension function takes values in the set $\{0, 1, 2, \dots, \aleph_0, 2^{\aleph_0}\}$, and corresponding to Theorem A it obeys

$$\dim(E \otimes F) = \dim E + \dim F$$

where $\otimes$ denotes the natural tensor product in the category of product systems. The relation of d_* to dim is given by the expected formula

$$d_*(\alpha) = \dim(E_\alpha).$$

It follows that, in order to classify E_0-semigroups up to cocycle conjugacy, one should seek to determine the structure of product systems. In particular, we may consider the set Σ of all isomorphism classes of product systems. The class of a product system E will be denoted $[E]$. There is a natural "addition" in Σ, defined by the natural tensor product operation

$$[E] + [F] = [E \otimes F],$$

which makes Σ into an abelian semigroup. There is a neutral element, which arises from the *trivial* product system Z. Z is defined as the trivial family of one-dimensional Hilbert spaces

$$Z = \{(t, z) : t > 0, z \in \mathbb{C}\ \}$$

where the inner product in $\mathbb{C}$ is the usual one $< z, w >= z\overline{w}$. The multiplication in Z is given by

$$(s, z) \cdot (t, w) = (s + t, zw).$$

It can be seen that if α is an E_0-semigroup which is trivial in the sense that each α_t is an automorphism, then E_α is isomorphic to Z. Moreover, it is also a fact that there are no nontrivial line bundles in Σ: every product system E with one-dimensional fiber spaces $E_t, t > 0$, is isomorphic to Z [6]. One can verify directly that for every $[E] \in \Sigma$ one has

$$[E \otimes Z] = [Z \otimes E] = [E].$$

Therefore $[Z]$ functions as an additive zero for Σ.

There is also a natural involution in Σ, defined by $[E] \mapsto [E^o]$, where E^o is the product system *opposite* to E. The structure of E^o is identical to that of E except that multiplication is reversed. With this involution, Σ becomes an abelian involutive semigroup with a zero element. *The problem of classifying E_0-semigroups up to cocycle conjugacy becomes the probem of determining the structure of the involutive semigroup Σ.*

At this point, we are not even certain of the *cardinality* of Σ! It is expected that Σ is uncountable, but this has not been proved. Notice for example that by the more general version of Theorem A alluded to above, the dimension function defines a homomorphism of Σ into the additive semigroup of extended nonnegative integers $\{0, 1, 2, \dots, \aleph_0, 2^{\aleph_0}\}$. Little is known about the quotient structure in Σ defined by this homomorphism. For example, by recent work of Powers in which a new family of E_0-semigroups is constructed [21], we now know that for each $k = 1, 2, \dots, \aleph_0$ there are infinitely many elements x of Σ that satisfy

$$\dim(x) = k.$$

But it is still not known if there are distinct elements $x, y \in \Sigma$ satisfying $\dim(x) = \dim(y) = 0$. Equivalently, is there a nontrivial E_0-semigroup α with the property that there is a nonzero unit $U = \{U_t : t \geq 0\}$ and such that every other unit V is related to U by a relation of the form

$$V_t = e^{i\lambda t} U_t, \qquad t \geq 0$$

where λ is a complex number?

Finally, let us return to some questions raised in section 1 concerning the problem of extending E_0-semigroups to automorphism groups acting on a larger type I factor. Suppose that we are given a *pair* of E_0-semigroup s α, β acting respectively on $\mathcal{B}(H)$ and $\mathcal{B}(K)$. We are interested in obtaining conditions on the pair α, β which imply that there is a one-parameter group of unitaries $W = \{W_t :\in \mathbb{R}\}$ acting on the tensor product $H \otimes K$ whose associated automorphism group $\gamma_t(C) = W_t C W_t^*$ satisfies

$$\begin{aligned} \gamma_t(A \otimes \mathbf{1}) &= \alpha_t(A) \otimes \mathbf{1}, \quad \text{for } t \geq 0 \\ \gamma_t(\mathbf{1} \otimes B) &= \mathbf{1} \otimes \beta_{-t}(B), \quad \text{for } t \leq 0. \end{aligned}$$

In case such a group exists, then α and β are said to be *paired.* This relation was introduced by Powers and Robinson in [22] as an intermediate step in their definition of another index. The Powers-Robinson index is an equivalence relation defined in the class of all E_0-semigroups; α and β are said to have the *same index* if there is a third E_0-semigroup σ with the property that α can be paired with σ and σ can be paired with β.

Using the theory of product systems one can determine the precise nature of this pairing, and thus give a more concrete form to the Powers-Robinson index. The details are as follows. It is not hard to show that α and β are paired iff their product systems E_α and E_β are *anti* isomorphic. Moreover, with any particular anti isomorphism $\theta : E_\alpha \to E_\beta$ one can write down a *specific* one parameter unitary group W acting on $H \otimes K$ which simultaneously extends α and β in the above sense (the details can be found in [2, pp 27–28]).

Carrying this one step further, we can answer a question posed in section 1 which asks how to describe the possible ways of extending an E_0-semigroup to a larger type I factor. More precisely, starting with a particular E_0-semigroup α we seek to describe all possible ways of finding a one-parameter unitary group $W = \{W_t : t \in \mathbb{R}\}$ acting on some other Hilbert space K and a type I subfactor $\mathcal{M} \subseteq \mathcal{B}(K)$ with the property that

$$W_t \mathcal{M} W_t^* \subseteq \mathcal{M}, \qquad \text{for } t \geq 0$$

and such that α is conjugate to the restriction of $\mathrm{ad}W_t, \quad t \geq 0$ to $\mathcal{M}$. The preceding remarks show that one should begin by considering the product system E_α^o opposite to E_α. Notice that by Theorem C, there exist E_0-semigroups whose product systems are isomorphic to E_α^o. Moreover, the set of all possible extensions of α is described by the set of all anti isomorphisms of E_α to E_α^o. In turn, these are obtained by composing the natural anti isomorphism of E_α to E_α^o with an arbitrary automorphism of E_α itself. Thus the set of all possible extensions of α is parameterized by the group of all automorphisms of the product system E_α. This group is computed explicitly for the "standard" examples in ther last section of [2]; its structure in the case of general product systems remains mysterious.

In particular, it follows from these remarks that two E_0-semigroups have the same index in the sense of Powers-Robinson iff their product systems determine the same element of the semigroup Σ. Thus, this discussion also gives a somewhat more concrete description of the Powers-Robinson index: it is now identified with this Σ-valued index map

$$\alpha \to [E_\alpha] \in \Sigma.$$

5. CCR **flows.**

We have already remarked that there is a sequence of *standard* E_0-semigroups, and corresponding to them a sequence of *standard* product systems. These have been described in detail in [2] and [7]. The purpose of this section is to give a brief description of these standard examples and to describe recent results on the problem of characterizing E_0-semigroups that are cocycle conjugate to a standard one.

It is useful to think of this construction as a functor related to second quantization; that interpretation makes explicit the precise sense in which the index $d_*(\alpha)$ of an E_0-semigroup is a quantized form of the Fredholm index of certain differential operators. A fuller discussion of these issues can be found in [7].

Consider the category $\mathcal{S}$ whose objects are semigroups of isometries $U = \{U_t : t \geq 0\}$ each of which acts on a separable Hilbert space H_U. $\hom(U, V)$ consists of unitary operators $W : H_U \to H_V$ which intertwine U and V:

$$WU_t = V_t W, \qquad t \geq 0.$$

This category admits a natural direct sum operation $\oplus$, in which $U \oplus V$ is the semigroup of isometries on $H_U \oplus H_V$ defined by

$$(U \oplus V)_t = U_t \oplus V_t, \qquad t \geq 0.$$

Every semigroup of isometries U decomposes uniquely into a direct sum

$$U = V \oplus W \tag{5.1}$$

where W is a semigroup of unitary operators and where V is *pure* in the sense that

$$\bigcap_{t>0} V_t H_V = 0.$$

Moreover, every pure semigroup of isometries is isomorphic to a direct sum of a countable number d of copies of the simple *shift* semigroup S which acts on $L^2(0, \infty)$ by way of

$$S_t f(x) = \begin{cases} f(x-t), & \text{for } x > t \\ 0, & \text{for } 0 < x \leq t. \end{cases}$$

The number d of copies of S is uniquely determined by V, and is called the *index* of V. We remark that there are other ways to define the index of V, involving the deficiency spaces of the infinitesimal generator of V. But the definition we have given is the quickest. All of this information about the decomposition (5.1) is often referred to as the *Wold decomposition*.

The index obeys the expected law of addition

$$\operatorname{ind}(U_1 \oplus U_2) = \operatorname{ind}(U_1) + \operatorname{ind}(U_2),$$

and it can take on any of the values $0, 1, 2, \dots, \aleph_0$. Notice that there is also a tensor product operation defined on $\mathcal{S}$, but it has terrible arithmetic properties with respect to the index. For example, if U_1 and U_2 have index 1 then $U_1 \otimes U_2$ has index $\aleph_0$. Thus we must consider $\mathcal{S}$ as a category with a single operation $\oplus$.

Let $\mathcal{E}$ be the category whose objects are E_0-semigroups and whose maps are conjugacies. Thus, if α and β are E_0-semigroups acting respectively on $\mathcal{B}(H_\alpha)$ and $\mathcal{B}(H_\beta)$, then $\hom(\alpha, \beta)$ consists of $*$-isomorphisms $\theta : \mathcal{B}(H_\alpha) \to \mathcal{B}(H_\beta)$ satisfying

$$\theta(\alpha_t(A)) = \beta_t(\theta(A))$$

for every $t \geq 0$, $A \in \mathcal{B}(H_\alpha)$. The natural operation in $\mathcal{E}$ is the tensor product of E_0-semigroups that has already been defined in section 3.

One might well ask if there is a direct sum operation in $\mathcal{E}$. Assuming that there were such an operation, one would expect $\alpha \oplus \beta$ to be an E_0-semigroup acting on $\mathcal{B}(H_\alpha \oplus H_\beta)$. By replacing β with a conjugate copy of itself if necessary, we may assume that $H_\alpha = H_\beta = H$. Then $\mathcal{B}(H \oplus H)$ consists of 2×2 matrices over $\mathcal{B}(H)$. One would expect *at least* that $\alpha \oplus \beta$ should restrict to α and β on the appropriate summands, that is $\alpha \oplus \beta$ should also have the property

$$(\alpha \oplus \beta)_t : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \to \begin{pmatrix} \alpha_t(A) & 0 \\ 0 & \beta_t(B) \end{pmatrix}, \tag{5.2}$$

for all $t \geq 0$ and all $A, B \in \mathcal{B}(H)$. However, if there were an E_0-semigroup on $\mathcal{B}(H \oplus H)$ which satisfied (5.2), then one could show that α and β must in fact be cocycle perturbations of each other. This argument is a variation of Connes' elegant observation about exterior equivalence of modular automorphism groups (the idea can be found in Lemma 8.11.2 of [17]). In particular, if α and β are not cocycle conjugate then it is impossible to make a reasonable definition of $\alpha \oplus \beta$. One might summarize this state of affairs as follows: *the only appropriate operation in the category $\mathcal{S}$ is the direct sum and the only possible operation in the category $\mathcal{E}$ is the tensor product.*

Finally, the index is defined on the objects of $\mathcal{E}$ and because of Theorem A we have

$$d_*(\alpha \otimes \beta) = d_*(\alpha) + d_*(\beta).$$

We now describe a construction which can be considered a functor from $\mathcal{S}$ to $\mathcal{E}$. This is (Boson) quantization, and it is also a form of exponentiation in that direct sums carry over to tensor products. The details are as follows.

Let H be a Hilbert space. We will write H^n for the symmetric tensor product of n copies of H if $n \geq 1$; H^0 is defined as $\mathbb{C}$. Let

$$e^H = \sum_{n=0}^{\infty} H^n$$

be the symmetric Fock space over H. The natural exponential map $\exp : H \to e^H$ is defined by

$$\exp(\xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \xi^{\otimes n}.$$

e^H is spanned by the vectors of the form $\exp(\xi)$, $\xi \in H$, and we have

$$< \exp(\xi), \exp(\eta) >= e^{<\xi,\eta>}.$$

For every $\xi \in H$ there is a unique unitary operator $W(\xi)$ on e^H which satisfies

$$W(\xi) \exp(\eta) = e^{-\frac{1}{2}\|\xi\|^2 - <\eta,\xi>} exp(\eta + \xi).$$

W is strongly continuous, obeys Weyl's form of the canonical commutation relations

$$W(\xi)W(\eta) = e^{i\mathrm{Im}<\xi,\eta>} W(\xi + \eta),$$

and $W(H)$ is an irreducible set of operators on e^H.

Now let $U \in \mathcal{S}$. We define an E_0-semigroup α^U on $\mathcal{B}(e^H)$ as follows. Fix $t \geq 0$. Because of the irreducibility of W, it is not hard to verify that there is a unique normal endomorphism α_t^U of $\mathcal{B}(e^H)$ satisfying

$$\alpha_t^U(W(\xi)) = W(U_t \xi), \qquad \xi \in H_U.$$

$\alpha^U = \{\alpha_t^U : t \geq 0 \}$ is an E_0-semigroup. Because of the natural identification

$$e^{H_1 \oplus H_2} = e^{H_1} \otimes e^{H_2}$$

we have a natural identificaiton

$$\alpha^{U \oplus V} = \alpha^U \otimes \alpha^V \tag{5.3}$$

for every $U, V \in \mathcal{S}$. With these observations in hand, one can establish the functoriality of the map $U \to \alpha^U$ (see [7] for more detail).

Let $U \in \mathcal{S}$. Applying (5.3) and the Wold decomposition (5.1), we find that α^U decomposes into a tensor product

$$\alpha^U = \beta \otimes \gamma$$

where γ is a trivial E_0-semigroup (i.e., each γ_t is an automorphism) and where β is conjugate to an E_0-semigroup of the form $\alpha^{d \cdot S}$ where $d \cdot S$ is a direct sum of d copies of the simple shift semigroup S acting on $L^2(0, \infty)$. An E_0-semigroup such as β is called a *CCR flow of index d*. This terminology is justified by the following index theorem [2],[7].

Theorem D.

$$d_*(\alpha^U) = \text{index}(U).$$

The proof of Theorem D involves some work: one must find *all* the units of α^U and compute the associated covariance function in order to calculate the dimension of its associated Hilbert space. It follows immediately from Theorem B that if α is any E_0-semigroup and γ is a *trivial* E_0-semigroup, then $\alpha \otimes \gamma$ is cocycle conjugate to α (actually, it is not hard to prove the latter directly) . If we collect this observation together with Theorem D then we are led to the conclusion that for any $U, V \in \mathcal{S}$, α^U is cocycle conjugate to α^V iff α^U and α^V have the same index. Thus the numerical index is a complete cocycle conjugacy invariant for these examples.

We also point out that there is a corresponding construction of "standard" examples of product systems which parallels what we have done for E_0-semigroups, and we refer the reader to [2] for the details.

Finally, one may use the Fermionic Fock space to construct standard examples of E_0-semigroups having index $1, 2 \dots, \aleph_0$ in a way that is roughly parallel to what we have done above (though we point out that the construction of the corresponding product systems is not so explicit in the Fermionic setting). The E_0-semigroups obtained from either the Bosonic or the Fermionic construction turn out to be conjugate, provided that their numerical index is the same.

6. Classification results.

In this section we will describe two characterizations of product sytems which are isomorphic to one of the standard product systems. These results provide a classification of E_0-semigroups which have sufficiently many units, or which have enough decomposable operators.

Let $\alpha = \{\alpha_t : t \geq 0 \}$ be an E_0-semigroup acting on $\mathcal{B}(H)$, and fix $t > 0$. Consider the set $\mathcal{S}_t$ of all operators which can be decomposed into a finite product of the form

$$T = U_1(t_1)U_2(t_2) \dots U_n(t_n), \tag{6.1}$$

where the $U_1, U_2, \dots, U_n$ are units for α, where $t_1, t_2, \dots, t_n$ are positive real numbers summing to t, and where n is an arbitrary positive integer. Because $t_1 + t_2 +$

$\cdots + t_n = t$ it is clear that $\mathcal{S}_t \subseteq E_\alpha(t)$ for every $t > 0$. We say that α is *completely spatial* if there is a $t > 0$ such that

$$H = \overline{\operatorname{span}}\{T\xi : T \in \mathcal{S}_t, \xi \in H\}. \tag{6.2}$$

It is easy to see that if (6.2) is true for some particular positive t, then it is true for every positive t.

It was proved in ([2] section 7) that every completely spatial E_0-semigroup is cocycle conjugate to a CCR flow. It follows that completely spatial E_0-semigroups are classified by their numerical index. This is proved at the level of product systems, using Theorem B.

That result has recently been extended significantly [9]. The extended version does not assume the existence of units, and is formulated as follows. Fix $t > 0$. An operator $T \in E_\alpha(t)$ is called *decomposable* if, for every $0 < s < t$ there are operators $A_s \in E_\alpha(s)$, $B_s \in E_\alpha(t-s)$ such that

$$T = A_s B_s.$$

We write $\mathcal{D}_t$ for the set of all decomposable operators in $E_\alpha(t)$. Any operator of the form (6.1) is decomposable because of the semigroups property of each U_k, and therefore $\mathcal{D}_t$ contains $\mathcal{S}_t$. In fact, it is not hard to see that the following conditions are equivalent

$$E_\alpha(t) = \overline{\operatorname{span}}\,\mathcal{D}_t \tag{6.3.1}$$

$$H = \overline{\operatorname{span}}\{T\xi : T \in \mathcal{D}_t, \xi \in H\}, \tag{6.3.2}$$

and that if the conditions (6.3) are satisfied for some particular t then they are satisfied for every positive t. We remark that one uses the Hilbert space topology on $E_\alpha(t)$ in (6.3.1); hence it is apparent that the conditions (6.3) depend only on the structure of the product system E_α associated with α. The main result of [9] is

Theorem E. *Every E_0-semigroup satisfying the conditions (6.3) is cocycle conjugate to a CCR flow.*

Utilizing an ingenious construction in [19], Powers showed that there are E_0-semigroups which possess no units whatsoever. In a recent paper [21] he also proved that there are E_0-semigroups which have units but which are *not* cocycle conjugate to a CCR flow. It follows that there are product systems which a) have no units, and others which b) have units but not enough units to generate the product system.

We believe that it should be possible to give a more direct consturction of product systems with these properties. Unfortunately, we do not yet know how to do this. There are examples of product systems that arise naturally in probability theory (see [2], pp 14–16). Some of these examples do not *appear* to contain enough units. However, in all such cases we have studied we eventually found many units that were not initially obvious. What *is* immediately obvious in these probabilistic examples is that the product systems are generated by their decomposable vectors. Theorem E tells us that such product systems must be standard ones. In particular, any attempt to construct nonstandard product systems from "decomposable" sets must fail.

7. Spectral invariant.

Theorem C above asserts that every abstract product system is associated with an E_0-semigroup. This result is analogous to the fact that every locally compact group G has a faithful unitary representation on a Hilbert space. The proof of the latter assertion about groups follows from an analysis of the properties of the group C^*-algebra $C^*(G)$, together with the Gelfand-Neumark theorem. In that result, $C^*(G)$ functions as the "spectrum" of the group G.

In this section we show how, starting with a product system E, one can construct a *spectral* C^*-algebra $C^*(E)$. Results like Theorem C are obtained by exploiting the properties of $C^*(E)$. More generally, $C^*(E)$ provides a "topological" invariant that is important for understanding the nature of product systems and their associated E_0-semigroups.

Let $p : E \to (0, \infty)$ be a product system, and let us write $E(t) = p^{-1}(t)$ for the Hilbert space over $t > 0$. We form the Hilbert space of L^2 sections

$$L^2(E) = \int_{(0,\infty)}^{\oplus} E(t)\, dt \ .$$

The inner product in $L^2(E)$ is the natural one

$$< \xi, \eta >= \int_0^\infty < \xi(t), \eta(t) > \ dt \ .$$

Let $f \in L^1(E)$ be an integrable section. Using the multiplication in E we see that for every $\xi \in L^2(E)$ and every $0 < t < x$,

$$f(t)\xi(x - t) \in E(x),$$

and hence we can define a measurable section $f * \xi$ by

$$f * \xi(x) = \int_0^x f(t)\xi(x - t)\, dt \ .$$

For fixed $f \in L^1(E)$, left convolution by f, $\xi \mapsto f * \xi$, defines a bounded linear operator on $L^2(E)$ of norm at most $\|f\|_1$. This operator is denoted l_f. A straightforward computation shows that for any two functions $f, g \in L^1(E)$, there are functions $h_1, h_2 \in L^1(E)$ such that

$$l_f^* l_g = l_{h_1} + l_{h_2}^*.$$

It follows that the linear span of all products of the form $l_f l_g^*$ is a self-adjoint subalgebra of $\mathcal{B}(L^2(E))$. $C^*(E)$ is defined as the norm-closure of this algebra

$$C^*(E) = \overline{\text{span}}\{l_f l_g^* : f, g \in L^1(E)\ \}. \tag{7.1}$$

The funcamental property of $C^*(E)$ is that its representations correspond to all possible E_0-semigroups α for which E_α is isomorphic to E. This is a key result in the theory and we want to state it precisely. It is convenient to slightly generalize the notion of E_0-semigroup. By an e_0-semigroup we mean a semigroup $\alpha = \{\alpha_t : t \geq 0\}$

of normal $*$-endomorphisms of $\mathcal{B}(H)$ that satisfies all of the conditions of an E_0-semigroup except that $\alpha_t(\mathbf{1})$ is not required to be $\mathbf{1}$. Thus, for an e_0-semigroup α,

$$P_t = \alpha_t(\mathbf{1})$$

defines a strongly continuous family of projections which decreases as $t \to \infty$. The limit

$$P_\infty = \lim_{t\to\infty} \alpha_t(\mathbf{1})$$

is an α-invariant projection which induces a decomposition of the underlying Hilbert space

$$H = H_\infty \oplus H_0,$$

where H_∞ and H_0 are, respectively, the ranges of the projections P_∞ and $\mathbf{1} - P_\infty$. The restriction of α to $\mathcal{B}(H_\infty)$ is an E_0-semigroup, and the restriction α^0 of α to $\mathcal{B}(H_0)$ is an e_0-semigroup whose limiting projection is zero:

$$\lim_{t\to\infty} \alpha^0_t(\mathbf{1}) = 0. \tag{7.2}$$

To this extent the study of e_0-semigroups reduces to the study of E_0-semigroups and the extreme case of e_0-semigroups satisfying (7.2).

We need to relate the representations of $C^*(E)$ more directly to E. By a *representation* of E we mean a measurable mapping $\phi : E \to \mathcal{B}(H)$ which restricts to a linear map on each fiber $E(t), t > 0$, which preserves multiplication in that $\phi(u)\phi(v) = \phi(uv)$ for all $u, v \in E$, and which obeys the following partial "commutation relation" on each fiber:

$$\phi(v)^*\phi(u) = < u, v > \mathbf{1}, \qquad u, v \in E(t). \tag{7.3}$$

If $\phi : E \to \mathcal{B}(H)$ is an arbitrary representation then we can define an e_0-semigroup $\alpha = \{\alpha_t : t > 0\}$ which acts on $\mathcal{B}(H)$ as follows. For each positive t, choose an orthonormal basis $\{e_1(t), e_2(t), \dots\}$ for $E(t)$ and put

$$\alpha_t(A) = \sum_{n=1}^{\infty} \phi(e_n(t))A\phi(e_n(t))^*, \tag{7.4}$$

$A \in \mathcal{B}(H)$. We define $\alpha_0(A) = A$. The left side is independent of the particular choice of basis $\{e_n(t)\}$, and it is true (though nontrivial) that α is an e_0-semigroup whose canonical product system is isomorphic to E. Indeed,

$$E_\alpha(t) = \{\phi(v) : v \in E(t)\}$$

and ϕ itself implements the stated isomorphism of E and $E\alpha$. These things are proved in [3].

More significantly, any representation $\phi : E \to \mathcal{B}(H)$ determines a unique representation $\pi : C^*(E) \to \mathcal{B}(H)$ by way of

$$\pi(l_f l_g^*) = \phi(f)\phi(g)^*, \qquad f, g \in L^1(E) \tag{7.5}$$

where for $f \in L^1(E)$, $\phi(f)$ denotes the operator integral

$$\phi(f) = \int_0^\infty \phi(f(x))\,dx.$$

The key property of $C^*(E)$ is that this association $\phi \to \pi$ is in fact a bijection [3]:

Theorem F. *The nondegenerate separable representations of $C^*(E)$ correspond bijectively with all e_0-semigroups α for which E_α is isomorphic to E.*

Because of Theorem F, we are let to examine the structure of $C^*(E)$, and attempt to describe its state space in terms that are as explicit as possible. The remainder of this paper is devoted to a discussion of progress on these issues.

The principal result of [3] is that $C^*(E)$ is a simple C^*-algebra in most (and perhaps all) cases. More precisely,

Theorem G. *For every product system E, $C^*(E)$ is a unitless nuclear C^*-algebra. If E possesses a nonzero unit, then $C^*(E)$ has no closed nontrivial ideals.*

In particular, the C^*-algebras $C^*(E_n), n = 1, 2, \dots, \aleph_0$ associated with the standard examples E_n (i.e., the product systems of the CCR flows) are all simple. These C^*-algebras are most like continuous versions of the Cuntz algebras $\mathcal{O}_n$, $n = 2, 3, \dots, \infty$ (see [3], [10]).

We still do not know if $C^*(E)$ is simple in the cases where $\mathcal{U}_E = \emptyset$. However, we do have the following information in general. Consider the one-parameter unitary group W defined on $L^2(E)$ by

$$W_t\xi(x) = e^{itx}\xi(x).$$

The generator N of W,

$$W_t = e^{itN}$$

replaces the *number operator* on ordinary Fock space. N has Lebesgue spectrum distributed throughout $[0, \infty)$ with infinite multiplicity. The associated one parameter group of automorphisms leaves $C^*(E)$ invariant,

$$W_t C^*(E) W_t^* = C^*(E), \qquad t \in \mathbb{R}$$

and thus induces a natural one parameter group of automorphisms $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of $C^*(E)$. γ is called the *gauge group*. The best general result in this direction is the following, from which Theorem G is easily deduced (see [3]).

Theorem G1. *There are no closed proper ideals J in $C^*(E)$ which are gauge invariant in the sense that $\gamma_t(J) = J$, for every $t \in \mathbb{R}$.*

Theorem F tells us that in order to specify an e_0-semigroup whose product system is isomorphic to E, it is enough to specify a state of $C^*(E)$. However, there remains a significant question: how does one know when a representation gives rise to an E_0-semigroup rather than, say, merely an e_0-semigroup? In order to discuss this, let us say that a representation π of $C^*(E)$ is *essential* if it gives rise to an E_0-semigroup, and *singular* if it gives rise to an e_0-semigroup satisfying 7.2. Similarly, a state of $C^*(E)$ (i.e., a nonzero positive linear functional on $C^*(E)$ with no condition on its norm) is called essential or singular according as the representation it defines via the GNS construction has the corresponding property.

The state space $\mathcal{P}$ of $C^*(E)$ is a norm-closed cone, and it is known that this cone decomposes into a direct of order ideals

$$\mathcal{P} = \mathcal{E} \oplus \mathcal{S}$$

where $\mathcal{E}$ (resp. $\mathcal{S}$) denotes the set of essential (resp. singular) states [4]. A detailed description of this decomposition and the singular summand $\mathcal{S}$ is given in [4]. Here, we want to concentrate on the description of the essential summand. In view of the precdding discussion, the assertion that every product system E is associated with an E_0-semigroup becomes the assertion that $C^*(E)$ has (nonzero) essential states...i.e., that $\mathcal{E} \neq \emptyset$. Our main result along these lines is the following result from [5] which will be discussed further in the following section.

Theorem H. *For every product system E, there is an essential state of $C^*(E)$ whose E_0-semigroup α is ergodic in the sense that*

$$\{A \in \mathcal{B}(H_\alpha) : \alpha_t(A) = A, \quad \forall t \geq 0\ \} = \mathbb{C}\mathbf{1}.$$

Because of Theorem F we have $E_\alpha \cong E$. This result is analogous to the fact that a locally compact group has irreducible unitary representations. Indeed, the relation that exists between a representation $\pi : C^*(E) \to \mathcal{B}(H)$ and its associated e_0-semigroup $\alpha = \{\alpha_t : t \geq 0\ \}$ acting on $\mathcal{B}(H)$ is expressed in (7.4) and (7.5). From the nature of this relation it follows that the fixed algebra of α,

$$\{A \in \mathcal{B}(H) : \alpha_t(A) = A \quad \forall t \geq 0\ \}$$

is precisely the commutant of $\pi(C^*(E)$. Thus, *the proof of Theorem G amounts to showing that $C^*(E)$ has nonzero essential pure states.* Such states will be discussed in the following section.

There are other consequences one can obtain along similar lines. For example, one knows that the C^*-algebra$C^*(E)$ is not GCR, and hence it has representations which generate factors of type II_∞ or III. There are even essential states with these properties, and hence we may conclude that every E_0-semigroup is cocycle conjugate to an E_0-semigroup whose fixed agebra is a factor of type II or III.

8. States in the regular representation.

$C^*(E)$ is defined as a C^*-algebra of operators on the Hilbert space $L^2(E)$, and this gives rise to a representation $\lambda : E \to \mathcal{B}(L^2(E))$. For $v \in E(t)$, $\lambda(v)$ is defined as

$$\lambda(v)\xi(x) = \begin{cases} v \cdot \xi(x-t), & \text{if } x > t \\ 0, & \text{if } 0 < t < x. \end{cases}$$

Notice that for $f \in L^1(E)$ we have

$$l_f = \int_0^\infty \lambda(f(x))\, dx.$$

λ is called the (left) regular representation. The associated e_0-semigroup α is singular, since $\alpha_t(\mathbf{1})$ is the projection onto to the subspace $\{\xi \in L^2(E) : \xi(x) = 0, \text{for } 0 < x \leq t\ \}$ and these subspaces decrease to 0 as $t \to \infty$. Actually, we will be more concerned with the e_0-semigroup generated by the right regular anti representation $\rho : E \to \mathcal{B}(L^2(E))$, where for $v \in E(t)$, $\rho(v)$ is defined as the operator

$$\rho(v)\xi(x) = \begin{cases} \xi(x-t) \cdot v, & x > t \\ 0, & 0 < x \leq t. \end{cases}$$

Notice that ρ reverses multiplication in the sense that $\rho(uv) = \rho(v)\rho(u)$, for $v, v \in E$. Nevertheless, we can use ρ to define a second e_0-semigroup $\beta = \{\beta_t : t \geq 0\}$ by way of

$$\beta_t(A) = \sum_{n=1}^{\infty} \rho(e_n(t))A\rho(e_n(t))^*, \quad A \in \mathcal{B}(L^2(E))$$

for $t > 0$, $\{e_1(t), e_2(t), \dots\}$ being any orthonormal basis for $E(t)$, and where β_0 is defined as the identity endomorphism. It is true (and nontrivial) that β is an e_0-semigroup [3],[5].

The generator of β is defined as the limit in the strong operator topology

$$\delta(A) = \lim_{t \to 0+} \frac{1}{t}(A - \beta_t(A)),$$

A ranging over the set $\mathcal{D}(\delta)$ of all operators for which the indicated limit exists. It is convenient here to use a different sign in the definition of δ than that of section 2. $\mathcal{D}(\delta)$ is a unital $*$-subalgebra of $\mathcal{B}(L^2(E))$ and δ is an unbounded self adjoint derivation from $\mathcal{D}(\delta)$ into $\mathcal{B}(L^2(E))$. We will first give an alternate description of $C^*(E)$ in terms of δ. This description of $C^*(E)$ is of key importance.

For every $t \geq 0$, let P_t denote the projection onto the subspace

$$\{\xi \in L^2(E) : \xi(x) = 0 \quad \text{a.e., for } 0 < x \leq t\}.$$

Notice that $\alpha_t(\mathbf{1}) = \beta_t(\mathbf{1}) = P_t$, and that $\{P_t : t \geq 0\}$ is a strongly continuous family of projections that increases from $\mathbf{0}$ to $\mathbf{1}$ as t moves from 0 to ∞. An operator $A \in \mathcal{B}(L^2(E))$ is said to have *bounded support* if there is a $t > 0$ such that $A = P_tAP_t$, and we write

$$\mathcal{B}_0 = \bigcup_{t>0} P_t\,\mathcal{B}(L^2(E))\,P_t \tag{8.1}$$

for the $*$-algebra of all operators of bounded support.

Significantly, every operator in $\mathcal{B}_0$ belongs to the range of δ. Indeed, if $A = P_tAP_t$, then the integral defined by the strong limit

$$I(A) = \lim_{T \to \infty} \int_0^T \beta_s(A)\,ds$$

exists and obeys

$$\|I(A)\| \leq t, \tag{8.2.1}$$

$$\delta(I(A)) = A \tag{8.2.2}$$

(see [5], Theorem 2.2). Indeed, the restriction of $I(\cdot)$ to $P_t\mathcal{B}(L^2(E))P_t$ is a normal completely positive linear map for every $t > 0$.

We will also write

$$H_0 = \bigcup_{t>0} P_tL^2(E)$$

for the linear space of all vectors $\xi \in L^2(E)$ which have bounded support, and

$$\mathcal{K}_0 = \mathcal{B}_0 \cap \mathcal{K}$$

for the $*$-algebra of all compact operators of bounded support.

The following result characterizes $C^*(E)$ in terms of the derivation δ.

Theorem I. *The set $\mathcal{A}$ of all operators A in the domain of δ satisfying $\delta(A) \in \mathcal{K}_0$ is a $*$-algebra whose norm closure is $C^*(E)$.*

Recall that $C^*(E)$ is spanned by operators of the form $l(f)l(g)^*$ where f and g are arbitrary integrable sections of E. Notice that if ξ, η are elements of $L^2(E)$ which have bounded support, then we may consider ξ and η as elements of $L^1(E)$, and operators of the form $l(\xi)l(\eta)^*$ also span $C^*(E)$. Such an operator belongs to $\mathcal{A}$ and the following formula implies that $\delta(l(\xi)l(\eta)^*$ is a rank-one operator in $\mathcal{K}_0$,

$$\delta(l(\xi)l(\eta)^*) = \xi \otimes \overline{\eta}, \qquad \xi, \eta \in H_0, \tag{8.3}$$

$\xi \otimes \overline{\eta}$ denoting the operator

$$\zeta \in L^2(E) \mapsto < \zeta, \eta > \xi.$$

The proof of (8.3) can be found in ([5], p. 288).

Theorem I and formula (8.3) open the way to a very explicit description of the state space of $C^*(E)$, which we now describe. Let ω be a linear functional defined on the algebra $\mathcal{B}_0$ of all operators having bounded support. ω is called a *locally normal weight* if, for every $t > 0$, the restriction of ω to $P_t\mathcal{B}(L^2(E))P_t$ is a normal positive linear functional. Locally normal weights are generalizations of normal weights (more precisely, of noncommutative Radon measures). Indeed, if

$$\omega : \mathcal{B}(L^2(E))^+ \to [0, +\infty]$$

is a normal weight satisfying $\omega(P_t) < +\infty$ for every $t > 0$, then the restriction of ω to $\mathcal{B}_0$ is a locally normal weight. We emphasize, however, that *not every locally normal weight can be extended to a normal weight of* $\mathcal{B}(L^2(E))$ (see [5], appendix A for an example). Thus, locally normal weights are more general than normal weights.

Notice that $\beta_t(\mathcal{B}_0) \subseteq \mathcal{B}_0$ for every $t > 0$. Thus we can make the following

Definition 8.4. *A locally normal weight ω is called decreasing if for every $t \geq 0$ we have*

$$\omega(\beta_t(A^*A)) \leq \omega(A^*A), \qquad A \in \mathcal{B}_0.$$

ω is called invariant if equality holds for every $t > 0$ and every $A \in \mathcal{B}_0$.

We will call such an ω simply a *decreasing weight.* Let $\mathcal{W}$ denote the cone of all decreasing weights satisfying the growth condition

$$\sup_{t>0} \frac{1}{t}\omega(P_t) < \infty.$$

Noting that $P_t = \mathbf{1} - \beta_t(\mathbf{1})$, it is clear that an invariant weight ω satisfies

$$\omega(\beta_s(\mathbf{1}) - \beta_{s+t}(\mathbf{1})) = \omega(\mathbf{1} - \beta_t(\mathbf{1}))$$

for all $s, t > 0$, and from this it follows that there is a constant $c \geq 0$ such that

$$\omega(P_t) = c \cdot t.$$

In particular, *every invariant weight belongs to $\mathcal{W}$.*

We can now describe the state space of $C^*(E)$ (see [5], Theorems 4.15 and 5.7). Every locally normal weight ω defines a linear functional $d\omega$ on $\mathcal{A}$ by way of

$$d\omega(A) = \omega(\delta(A)).$$

Notice that $d\omega$ is the "derivative" of ω in the direction of the flow of the e_0-semigroup β. We need to know when $d\omega$ is a positive linear functional which has finite norm. The characterization is as follows.

Theorem J. *For every decreasing weight $\omega \in \mathcal{W}$, $d\omega$ is a positive linear functional on $\mathcal{A}$ having norm*

$$\|d\omega\| = \sup_{t>0} \frac{1}{t}\omega(P_t).$$

$\omega \to d\omega$ is an affine order isomorphism of the cone $\mathcal{W}$ onto the state space of $C^*(E)$, which maps the subcone of invariant weights onto the cone of essential states.

With Theorem J in hand, it is easy to show that $C^*(E)$ must have essential states. A straightforward construction allows one to write down an invariant weight ω on $\mathcal{B}_0$ which is normalized so that

$$\omega(P_t) = t, \qquad \forall t > 0$$

(see Theorem 5.9 of [5]). It follows from Theorem J that $d\omega$ is an essential state of $C^*(E)$ satisfying $\|d\omega\| = 1$.

Corollary. *For every abstract product system E, there is an E_0-semigroup α for which E_α is isomorphic to E.*

As we have pointed out previously, with a little care, one can arrange that α is ergodic in the sense that

$$\alpha_t(A) = A, \quad \forall t \geq 0 \implies A = \text{scalar}.$$

The details can be found in [5].

There are numerous interesting unsolved problems concerning the spectral C^*-algebras $C^*(E)$. For example, if E_n is the standard product system of dimension n, $n = 1, 2, \dots, \aleph_0$, then $C^*(E)$ is known t be a "continuous time" analogue of the Cuntz algebra $\mathcal{O}_{n+1}$ (see [10]). However, we do not yet know if these C^*-algebras $C^*(E_n)$ are mutually non-isomorphic for different values of n. Cerainly this is the case for the $\mathcal{O}_n$. In fact, Cuntz showed that $\mathcal{O}_m$ is not isomorphic to $\mathcal{O}_n$ essentially by calculating the K-theory of these C^*-algebras [15],[16]. But while the K-theory of the C^*-algebras $C^*(E_n)$ has not been computed, there is some evidence that K-theory may not be capable of distinguishing between them.

Finally, we want to point out the remarkable fact that, like $\mathcal{O}_\infty$, every spectral C^*-algebra $C^*(E)$ has an unbounded trace τ. This is easily seen using the description of $C^*(E)$ given in Theorem I. Let $\mathcal{A}_1$ denote the set of all operators A in the domain of δ such that $\delta(A)$ is a *trace class* operator of bounded support. $\mathcal{A}_1$ is a self-adjoint ideal in $\mathcal{A}$ which is clearly norm-dense in $C^*(E)$. We can define a linear functional τ on $\mathcal{A}_1$ by way of

$$\tau(A) = \text{trace}(\delta(A)).$$

Notice that $\tau(AB) = \tau(BA)$, since

$$\begin{aligned}\text{trace}(\delta(AB)) &= \text{trace}(A\delta(B)) + (\delta(A)B) = \text{trace}(\delta(B)A) + \text{trace}(B\delta(A)) \\ &= \text{trace}(\delta(BA)).\end{aligned}$$

It is possible to show that τ is *not* a positive trace in that there exist operators $A \in \mathcal{A}_1$ satisfying $\tau(A^*A) < 0$. Such traces are uncommon in operator theory, but notice that the Wodzicki residue provides another example of an unbounded non-positive trace on an algebra of pseudo-differential operators [14]. As yet, the role of this trace in the theory of E_0-semigroups remains mysterious.

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