DISCRETIZED CCR ALGEBRAS

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To the memory of John Bunce

Abstract. We discuss how the canonical commutation relations must be modified in order to make appropriate numerical models of quantum systems. The C^*-algebras associated with the discretized *CCRs* are the non-commutative spheres of Bratteli, Elliott, Evans and Kishimoto.

1991 *Mathematics Subject Classification*. Primary 46L40; Secondary 81E05.
Key words and phrases. C^*-algebras, commutation relations, quantum mechanics.
This research was supported in part by NSF grant DMS89-12362

1. Introduction.

We consider the problem of discretizing the Hamiltonian of a one-dimensional quantum system in a form that is appropriate for carrying out numerical studies. Specifically, we start with a formal Schrödinger operator

$$H = \frac{1}{2}P^2 + v(Q)$$

acting on the Hilbert space $L^2(\mathbb{R})$, where P and Q are the canonical operators

$$P = -i\frac{d}{dx}$$
$$Q = \text{multiplication by } x,$$

and v is a real-valued continuous function of a real variable. The problem of discretizing H is that of finding an approximation to H which satisfies two requirements: (a) the basic principles of numerical analysis are satisfied, and (b) the uncertainty principle is preserved.

In [1, §§ 1–2], we argued that in order to satisfy these two conditions one must first replace P, Q with the pair

$$P_\tau = \frac{1}{\tau}\sin(\tau P)$$
$$Q_\tau = \frac{1}{\tau}\sin(\tau Q).$$

Here, τ is a fixed positive real number, the numerical step size. The discretized Hamiltonian is then defined as the following bounded self-adjoint operator on $L^2(\mathbb{R})$:

$$H_\tau = \frac{1}{2}P_\tau^2 + v(Q_\tau).$$

Obviously, H_τ belongs to the unital C^*-algebra $C^*(P_\tau, Q_\tau)$ generated by P_τ and Q_τ. We show that when τ^2/π is irrational (e.g., when τ is a rational number), $C^*(P_\tau, Q_\tau)$ is isomorphic to the non-commutative sphere $\mathcal{B}_{\tau^2}$ of Bratteli, Evans, Elliott and Kishimoto [5][6]; hence it is a simple C^*-algebra with a unique trace. We also describe the way in which the canonical commutation relations must be "discretized" in order to accommodate pairs of operators (P_τ, Q_τ) of this type. Together, these observations serve to make a more philosophical point, namely *non-commutative spheres will arise in any serious attempt to model quantum systems on a computer.*

In the "linear" case where v has the form $v(x) = cx^2/2$, c being a positive constant, the operator H_τ turns out to be unitarily equivalent to an operator of the form $\lambda M + \mu I$, where λ and μ are real constants and M is the almost Mathieu Hamiltonian

$$M = U + U^* + c(V + V^*),$$

associated with a pair of unitary operators U, V satisfying

$$VU = e^{i4\tau^2}UV.$$

An extensive amount of work has been done to compute the spectra of such operators. Here, we mention only [2], [3], [4], [9], [16] and refer the reader to the monograph [8] for further references.

Finally, I would like to thank Larry Schweitzer for pointing out the references [11] and [13] (as well as the relevance of his own work [17]) in connection with the spectral invariance property of the Banach $*$-algebra $l^1(\mathbb{Z} \oplus \mathbb{Z}, \omega)$.

2. Discretized CCR algebras.

Let θ be a real number such that θ/π is irrational, and let ω be the bicharacter of the discrete abelian group $G = \mathbb{Z} \oplus \mathbb{Z}$ defined by

2.1. $$\omega((m,n),(p,q)) = e^{i(np-mq)\theta/2}$$

A uniformly bounded family $\{D_x : x \in G\}$ of self-adjoint operators on a Hilbert space H is said to satisfy the *discretized canonical commutation relations* if

2.2 $$D_x D_y = \omega(x,y) D_{x+y} + \omega(y,x) D_{x-y}, \qquad x, y \in G.$$

Remarks. Notice that (2.2) is a generalization of the elementary trigonometric identity

$$2\cos A \cos B = \cos(A+B) + \cos(A-B),$$

in which phase shifts have been added by way of the cocycle ω. Indeed, for any pair of real numbers α, β, the function $D : G \to \mathbb{R}$ defined by

$$D(m,n) = 2\cos(\alpha m + \beta n)$$

satisfies (2.2) for the trivial cocycle $\omega = 1$. It is related to formula (2.2) of [5], except that our operators are self-adjoint and the phase factor is associated with a *nondegenerate* bicharacter ω.

The purpose of this section is to associate a C^*-algebra with the relations (2.2), and to point out some of its basic properties. Let $\{D_x : x \in G\}$ satisfy (2.2). It is clear that the norm closed linear span

$$\mathcal{D} = \overline{span}\{D_x : x \in G\}$$

is a separable C^*-algebra. Thus by passing from H to the subspace $[\mathcal{D}H]$ if necessary, we can assume that $\mathcal{D}$ is nondegenerate.

Proposition 2.3.

(i) $D_0 = 2I$.
(ii) $D_{-x} = D_x$.
(iii) $\|D_x\| \leq 2$, *for every* $x \in G$.

proof. Setting $y = 0$ in (2.2) we obtain $D_x D_0 = 2D_x$ for all $x \in G$, from which (i) is evident. Setting $x = 0$ in (2.2) now leads to $2D_y = D_0 D_y = D_y + D_{-y}$, hence (ii). For (iii), let

$$M = \sup_{x \in G} \|D_x\|.$$

By hypothesis, $M < \infty$. Moreover, setting $y = x$ in (2.2) gives

$$D_x^2 = D_{2x} + D_0 = D_{2x} + 2I$$

and thus $M^2 \leq M + 2$. This inequality implies that $-1 \leq M \leq 2$, hence (iii) □

We now construct a Banach $*$-algebra whose representations are associated with operator realizations of (2.2). Let $l^1(G, \omega)$ denote the Banach space of all absolutely summable complex functions on G, endowed with the multiplication and involution

$$f * g(x) = \sum_y \omega(y,x) f(y) g(x-y)$$

$$f^*(x) = \overline{f(-x)}.$$

It is easily checked that the linear subspace

$$D_\theta = \{f \in l^1(G, \omega) : f(-x) = f(x), \quad x \in G\}$$

is in fact a $*$-subalgebra of $l^1(G, \omega)$. Of course, the adjoint operation in D_θ simplifies to $f^*(x) = \overline{f(x)}$. Moreover, D_θ is linearly spanned by the elements

$$d_x = \delta_x + \delta_{-x},$$

δ_x denoting the unit function supported at x, and one has

$$\begin{aligned} d_x d_y &= \omega(x, y) d_{x+y} + \omega(y, x) d_{x-y} \\ \|d_x\| &= 2 \\ d_x &= d_{-x} = d_x^*. \end{aligned}$$

Proposition 2.4. *Let $\{D_x : x \in G\}$ be a uniformly bounded family of self-adjoint operators on a Hilbert space H satisfying* (2.2)*. Then there is a unique representation $\pi : D_\theta \to \mathcal{B}(H)$ such that*

$$\pi(d_x) = D_x, \qquad x \in G.$$

proof. By proposition (2.3), we know that $\|D_x\| \leq 2$; hence

$$\pi(f) = \frac{1}{2} \sum_{x \in G} f(x) D_x$$

defines a contractive self-adjoint linear mapping of D_θ into $\mathcal{B}(H)$. Moreover, using (2.2) we have

$$\begin{aligned} \pi(f)\pi(g) &= \frac{1}{4} \sum_{x,y} f(x) g(y) (\omega(x, y) D_{x+y} + \omega(y, x) D_{x-y}) \\ &= \frac{1}{4} \sum_{z,x} f(x) g(z - x) \omega(x, z) D_z + \frac{1}{4} \sum_{z,x} f(x) g(x - z) \omega(-z, x) D_z. \end{aligned}$$

Using the fact that $g(x - z) = g(z - x)$ and $\omega(-z, x) = \omega(x, z)$, the right side becomes

$$\frac{1}{2} \sum_z (\sum_x f(x) g(z - x) \omega(x, z)) D_z = \pi(f * g),$$

as required.

Finally, taking $f = \delta_x + \delta_{-x} = d_x$ and using (ii) of (2.3), we find that

$$\pi(d_x) = \frac{1}{2}(f(x) D_x + f(-x) D_x) = D_x$$

as required □

Remarks. It follows that the enveloping C^*-algebra $C^*(D_\theta)$ is the universal C^*-algebra generated by the commutation relations (2.2).

Let α be an automorphism of the discrete abelian group $\mathbb{Z} \oplus \mathbb{Z}$. Then α is given by a 2×2 integer matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by way of $\alpha(m, n) = (am + bn, cm + dn)$, where $a, b, c, d \in \mathbb{Z}$ satisfy the condition

$$\det \alpha = ad - bc = \pm 1.$$

It follows that

$$\omega(\alpha x, \alpha y) = \begin{cases} \omega(x, y), & \text{if } \det \alpha = +1 \\ \omega(y, x), & \text{if } \det \alpha = -1. \end{cases}$$

Hence the group $SL(2, \mathbb{Z})$ of determinant 1 automorphisms acts naturally on D_θ (resp. $C^*(D_\theta)$) as a group of $*$-automorphisms. Any $\alpha \in aut(\mathbb{Z} \oplus \mathbb{Z})$ satisfying $\det \alpha = -1$ gives rise to a $*$-anti-automorphism of D_θ (resp. $C^*(D_\theta)$).

Finally, notice that there is a natural $*$-homomorphism which carries D_θ into the irrational rotation C^*-algebra $\mathcal{A}_\theta$. Indeed, D_θ is obviously contained in the larger Banach $*$-algebra $l^1(\mathbb{Z} \oplus \mathbb{Z}, \omega)$ obtained by simply dropping the requirement that $f(-x) = f(x)$. It is clear that $l^1(\mathbb{Z} \oplus \mathbb{Z}, \omega)$ is the universal Banach $*$-algebra generated by unitary operators $\{W_x : x \in \mathbb{Z} \oplus \mathbb{Z}\}$ satisfying

$$W_x W_y = \omega(x, y) W_{x+y}, \qquad x, y \in \mathbb{Z} \oplus \mathbb{Z}.$$

Because of the formula (2.1) giving ω in terms of θ, the unitary elements U, V defined by $U = W_{(1,0)}, V = W_{(0,1)}$ satisfy $VU = e^{i\theta} UV$, and of course they generate $l^1(\mathbb{Z} \oplus \mathbb{Z}, \omega)$ as a Banach $*$-algebra. It follows that the enveloping C^*-algebra of $l^1(\mathbb{Z} \oplus \mathbb{Z}, \omega)$ is $\mathcal{A}_\theta$. Thus we obtain a morphism of D_θ into $\mathcal{A}_\theta$ by simply restricting the completion map

$$\gamma : l^1(\mathbb{Z} \oplus \mathbb{Z}, \omega) \to \mathcal{A}_\theta$$

to D_θ. By the universal property of enveloping C^*-algebras there is correspondingly a unique morphism of C^*-algebras

$$\gamma_B : C^*(D_\theta) \to \mathcal{A}_\theta.$$

In the next section it will be shown that γ_B is injective and we will identify its range.

3. Spectral Invariance and Extensions of States.

Let A be a Banach $*$-algebra with unit, and let

$$A^+ = \overline{\{a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n : a_k \in A, n \geq 1\}}$$

denote the closed positive cone in A. For simplicity, we assume throughout this section that the completion map γ of A into its enveloping C^*-algebra is *injective*.

Let B be a unital self-adjoint Banach subalgebra of A. We are interested in determining whether or not the C^*-algebra obtained by closing $\gamma(B)$ in the norm of $C^*(A)$ is the enveloping C^*-algebra of B. More precisely, we seek conditions under which the $*$-homomorphism $\gamma_B : C^*(B) \to C^*(A)$ defined by the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\text{incl}} & A \\ \downarrow & & \downarrow \\ C^*(B) & \xrightarrow{\gamma_B} & C^*(A) \end{array} \tag{3.1}$$

should be *injective*. Elementary considerations show that the following three conditions are equivalent:

(1) γ_B is injective.
(2) Every positive linear functional on B can be extended to a positive linear functional on A.
(3) $A^+ \cap B \subseteq B^+$.

Note, for example, that the implication (3) $\implies$ (2) is the extension theorem of M. G. Krein [15, p. 227], whereas (2) $\implies$ (3) follows from a standard separation theorem. It is not hard to find examples showing that these conditions are not always satisfied (see Appendix).

A is said to have the *spectral invariance property* if for every element $a \in A$ which is invertible in $C^*(A)$, we have $a^{-1} \in A$. This is equivalent to the assertion that the spectrum of any element of A is the same whether it is computed in A or in $C^*(A)$, or that A is closed under the holomorphic functional calculus of $C^*(A)$ (see [10, p. 52] for further significant consequences of spectral invariance in more general Frèchet algebras).

A familiar Tauberian theorem of Wiener asserts that if a continuous function on the unit circle never vanishes and has an absolutely convergent Fourier series

$$f(e^{i\theta}) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta},$$

$\sum |a_n| < \infty$, then $1/f$ has an absolutely convergent Fourier series. Of course, this is precisely the assertion that the group algebra $l^1(\mathbb{Z})$ has the spectral invariance property. While this theorem has a simple proof using the Gelfand theory, it is certainly not a triviality.

The significance of spectral invariance for our purposes derives from the following.

Proposition 3.2. *Let A be a unital Banach $*$-algebra which admits spectral invariance. Then for every self-adjoint unital Banach subalgebra B of A, the natural $*$-homomorphism*

$$\theta_B : C^*(B) \to C^*(A)$$

is injective.

proof. We will verify property (3) above by showing that $A^+ \cap B \subseteq B^+$. We may clearly assume that $A \subseteq C^*(A)$, as a self-adjoint subalgebra which is a Banach algebra relative to a larger norm than that of $C^*(A)$.

Choose $x \in A^+ \cap B$; without loss of generality we may assume that the B-norm of x is less than 1. Since x belongs to the positive cone of $C^*(A)$ its spectrum in $C^*(A)$ is nonnegative. By spectral invariance we have $\sigma_A(x) \subseteq [0, 1)$. Moreover, since

$\sigma_A(x)$ cannot separate the complex plane, we see from the spectral permanence theorem that $\sigma_B(x) = \sigma_A(x) \subseteq [0, 1)$. Hence for sufficiently small ϵ we have $\sigma_B(x + \epsilon 1) \subseteq (\epsilon, 1)$. Thus we may apply the power series

$$\sqrt{t} = \sum_{n=0}^{\infty} a_n (1-t)^n, \qquad |1-t| < 1$$

to the element $x + \epsilon 1$ to obtain a square root in B, i.e., a self-adjoint element $h \in B$ satisfying $x + \epsilon 1 = h^2$. This shows that $x + \epsilon 1 \in B^+$, and we obtain the desired conclusion by allowing ϵ to tend to zero ☐

We now apply this to show that the enveloping C^*-algebra $C^*(D_\theta)$ is isomorphic to the non-commutative sphere $\mathcal{B}_\theta$ of [6]. If we realize the irrational rotation C^*-algebra $\mathcal{A}_\theta$ as the C^*-algebra generated by a pair of unitary operators U, V satisfying $VU = e^{i\theta}UV$, then there is a unique automorphism σ of $\mathcal{A}_\theta$ satisfying $\sigma(U) = U^{-1}$, $\sigma(V) = V^{-1}$. In case θ/π is irrational, $\mathcal{B}_\theta$ is defined to be the fixed subalgebra

$$\mathcal{B}_\theta = \{a \in \mathcal{A}_\theta : \sigma(a) = a\}.$$

Let $\{W_x : x \in \mathbb{Z} \oplus \mathbb{Z}\}$ be the family of unitary operators in $\mathcal{A}_\theta$ defined by

$$W_{(m,n)} = e^{imn\theta/2} U^m V^n, \qquad m, n \in \mathbb{Z}.$$

One verifies easily that

$$W_x W_y = \omega(x, y) W_{x+y},$$

where ω is the bicharacter on $\mathbb{Z} \oplus \mathbb{Z}$ defined in (2.1), and moreover the action of σ is given by

$$\sigma(W_x) = W_{-x}, \qquad x \in \mathbb{Z} \oplus \mathbb{Z}.$$

Since $\mathcal{A}_\theta$ is spanned by $\{W_x : x \in \mathbb{Z} \oplus \mathbb{Z}\}$, we conclude that $\mathcal{B}_\theta$ is spanned by $\{W_x + W_{-x} : x \in \mathbb{Z} \oplus \mathbb{Z}\}$.

Corollary. *Suppose θ is not a rational multiple of π, and let $\alpha : D_\theta \to \mathcal{A}_\theta$ be the morphism defined by*

$$\alpha(d_x) = W_x + W_{-x}, \qquad x \in \mathbb{Z} \times \mathbb{Z}.$$

Then the natural extension $\tilde{\alpha} : C^*(D_\theta) \to \mathcal{A}_\theta$ gives an isomorphism of C^*-algebras

$$C^*(D_\theta) \cong \mathcal{B}_\theta.$$

proof. Let $G = \mathbb{Z} \oplus \mathbb{Z}$ and let $\omega : G \times G \to \mathbb{T}$ be the bicharacter of (2.1). Consider the Banach $*$-algebra $l^1(G, \omega)$, where multiplication and involution are defined respectively by

$$f * g(x) = \sum_y \omega(y, x) f(y) g(y - x)$$
$$f^*(x) = \overline{f(-x)}.$$

Notice first that $C^*(G, \omega)$ is naturally identified with the irrational rotation C^*-algebra $A_\theta = C^*(U, V)$, where U and V are unitary operators satisfying the above relation $VU = e^{i\theta}UV$. Indeed, letting $\{W_x : x \in G\}$ be the operators of $\mathcal{A}_\theta$ defined in the preceding remarks, it is clear that we can define a morphism of $l^1(G, \omega)$ into $\mathcal{A}_\theta$ by

$$\gamma(\delta_x) = W_x, \qquad x \in G,$$

δ_x denoting the unit function at x. The range of γ is dense in $\mathcal{A}_\theta$, and the natural extension of γ to $C^*(l^1(G, \omega))$ is injective because of the familiar universal property of such pairs U, V.

D_θ is indentified (via an isometric isomorphism of Banach $*$-algebras) with a subalgebra of $l^1(G, \omega)$,

$$D_\theta = \{f \in l^1(G, \omega) : \sigma_0(f) = f\},$$

where σ_0 is the $*$-automorphism of $l^1(G, \omega)$ given by

$$\sigma_0(f)(x) = f(-x), \qquad x \in G.$$

It is clear that the restriction of γ to D_θ carries $d_x = \delta_x + \delta_{-x}$ to $W_x + W_{-x}$; hence by the preceding remarks $\gamma(D_\theta)$ is a dense $*$-subalgebra of $\mathcal{B}_\theta$. Thus γ extends naturally to a surjective $*$-homomorphism of $C^*(l^1(G, \omega))$ onto $\mathcal{B}_\theta$, and it remains only to show that the latter morphism is injective.

Now it is known that $l^1(G, \omega)$ admits spectral invariance (see [13, Satz 5] for example, or apply Theorem 1.1.3 of [17] together with the results of [11] on the symmetry of the group algebra of the rank 3 discrete Heisenberg group); hence Proposition 3.2 implies that $\gamma \restriction_{D_\theta}$ extends uniquely to a $*$-isomorphism of $C^*(D_\theta)$ onto $\overline{\gamma(D_\theta)} = \mathcal{D}_\theta$ □

4. Representations.

In this section we make some general comments about the representation theory of the discretized CCRs (2.2). We assume throughout that θ is a real number such that θ/π is irrational.

Remark 4.1: Finite representations.

The unique trace on the irrational rotation algebra $\mathcal{A}_\theta$ gives rise to a representation of $\mathcal{A}_\theta$ which generates the hyperfinite II_1 factor R. The closure of $\mathcal{B}_\theta$ in this representation is a sub von Neumann algebra of R. Since $\mathcal{B}_\theta$ has a unique tracial state [5], it follows that the closure of $\mathcal{B}_\theta$ is a subfactor of R, and hence is also isomorphic to R. Moreover, since $\mathcal{B}_\theta$ is also simple [5], any finite representation of $\mathcal{B}_\theta$ is quasi-equivalent to this one.

It is not hard to show that the subfactor of R generated by $\mathcal{B}_\theta$ in the above representation has Jones index 2. Since any two subfactors of R of index 2 are known to be isomorphic [12], we have here a very stable invariant for the embedding of the discretized CCR algebra in the irrational rotation algebra $\mathcal{A}_\theta$.

In particular, by the corollary of 3.2 we may conclude from these remarks that *there is a representation of the discretized $CCRs$* (2.2) *which generates R as a von Neumann algebra; moreover any finite representation of the discretized $CCRs$ is quasi-equivalent to this one.*

Now let τ be a positive real number such that τ^2/π is irrational, and let P_τ and Q_τ be the discretized canonical operators on $L^2(\mathbb{R})$ associated with the step size τ as in section 1. We want to make explicit the relation that exists between the pair (P_τ, Q_τ) and the C^*-algebra $C^*(\mathcal{B}_{\tau^2})$ discussed in section 2.

Theorem 4.2. *There is a unique representation π of D_{τ^2} on $L^2(\mathbb{R})$ satisfying*

$$\pi(d_{(1,0)}) = 2\tau Q_\tau,$$
$$\pi(d_{(0,1)}) = 2\tau P_\tau.$$

$\pi(D_{\tau^2})$ and $\{P_\tau, Q_\tau\}$ generate the same unital C^*-algebra. Thus, the three C^*-algebras

$$C^*(D_{\tau^2}), \quad \mathcal{B}_{\tau^2}, \quad C^*(P_\tau, Q_\tau)$$

are mutually isomorphic.

proof. Let U, V be the one-parameter groups

$$U_t f(x) = e^{itx} f(x),$$
$$V_t f(x) = f(x+t) \qquad f \in L^2(\mathbb{R}).$$

As in section 1 we have

4.3
$$Q_\tau = \frac{1}{2i\tau}(U_\tau - U_{-\tau}) = \frac{1}{\tau}\sin(\tau Q),$$
$$P_\tau = \frac{1}{2i\tau}(V_\tau - V_{-\tau}) = \frac{1}{\tau}\sin(\tau P).$$

We claim first that the *sines* in (4.3) can be replaced by *cosines* in the sense that the pair (P_τ, Q_τ) is unitarily equivalent to the pair $(\tilde{P}_\tau, \tilde{Q}_\tau)$ given by

4.4
$$\tilde{Q}_\tau = \frac{1}{2\tau}(U_\tau + U_{-\tau})$$
$$\tilde{P}_\tau = \frac{1}{2\tau}(V_\tau + V_{-\tau}).$$

To see this, put $\lambda = \pi/2\tau$ and let R denote the reflection on $L^2(\mathbb{R})$ given by $Rf(x) = f(-x)$. Consider the unitary operator

$$W = RU_{-\lambda}V_\lambda.$$

Using the commutation relations $V_t U_s = e^{ist} U_s V_t$ together with $RU_sR^* = U_{-s}$ and $RV_tR^* = V_{-t}$, one finds that

$$WU_sW^* = e^{i\lambda s} U_{-s}$$
$$WV_tW^* = e^{i\lambda t} V_{-t}.$$

Noting that $e^{i\lambda\tau} = \sqrt{-1}$, we obtain (4.4) by applying adW to (4.3), i.e.,

$$WQ_\tau W^* = \frac{1}{2\tau}(U_\tau + U_{-\tau}) = \frac{1}{2\tau}\cos(\tau Q)$$
$$WP_\tau W^* = \frac{1}{2\tau}(V_\tau + V_{-\tau}) = \frac{1}{2\tau}\cos(\tau P).$$

We may therefore assume that the pair (Q_τ, P_τ) is defined by (4.4).

For each $x = (m,n) \in \mathbb{Z} \oplus \mathbb{Z}$, define a unitary operator W_x by

$$W_{(m,n)} = e^{imn\tau^2/2} U_{m\tau} V_{n\tau}.$$

A straightforward computation shows that the family of unitaries $\{W_x : x \in \mathbb{Z} \oplus \mathbb{Z}\}$ satisfies

$$W_x W_y = \omega(x,y) W_{x+y}$$

ω being the cocycle of (2.1) for the value $\theta = \tau^2$, and hence there is a representation π of $l^1(\mathbb{Z} \oplus \mathbb{Z}, \tau^2)$ on $L^2(\mathbb{R})$ such that

$$\pi(w_x) = W_x, \qquad x \in \mathbb{Z} \oplus \mathbb{Z}.$$

It is clear that π carries $d_{(1,0)}$ (resp. $d_{(0,1)}$) to $U_\tau + U_{-\tau} = 2\tau Q_\tau$ (resp. $2\tau P_\tau$).

It remains to show that the restriction of π to $C^*(D_{\tau^2})$ is uniquely defined by its values on the two elements $d_{(1,0)}, d_{(0,1)}\}$, and that Q_τ and P_τ generate $\pi(C^*(D_{\tau^2}))$ as a unital C^*-algebra. We will prove both by showing that the two elements $\{d_{(1,0)}, d_{(0,1)}\}$ and the identity generate the Banach $*$-algebra D_{τ^2}. It is not hard to adapt the results of [5] to prove that these three elements generate D_{τ^2}. Instead, we present the following argument since it gives somewhat more structural information.

Actually, we will give a fairly explicit method for calculating each element $d_x = \delta_x + \delta_{-x}$ in terms of the self-adjoint elements $p = d_{(1,0)}$ and $q = d_{(0,1)}$, using a "generating function" for the family $\{d_x : x \in \mathbb{Z} \oplus \mathbb{Z}\}$. Indeed, it suffices to establish the following lemma.

Lemma 4.5. *Let θ be a real number such that θ/π is irrational, and consider the real-analytic function $F : (-1,1) \times (-1,1) \to D_\theta$ defined by*

4.6
$$F(s,t) = \sum_{m,n=-\infty}^{+\infty} s^{|m|} t^{|n|} e^{-imn\theta/2} d_{(m,n)}.$$

(i) *For $-1 < u < 1, -2 \le x \le 2$, let*

$$\phi(u,x) = \frac{1-u^2}{1+u^2-ux}.$$

Noting that ϕ is separately analytic in each variable, we have

$$F(s,t) = 2\phi(s,q)\phi(t,p), \qquad |s|, |t| < 1,$$

where q, p are the elements of D_θ defined by

$$q = d_{(1,0)}, \qquad p = d_{(0,1)}.$$

(ii) *The Banach $*$-algebra D_θ is spanned by the set $\mathcal{F} \cup \mathcal{F}^*$, where*

$$\mathcal{F} = \{F(s,t) : |s|, |t| < 1\}.$$

proof of (i) Let ω be the bicharacter of $\mathbb{Z} \oplus \mathbb{Z}$ defined by

$$\omega((p,q),(m,n)) = e^{i(qm-pn)\theta/2}$$

and let u, v be the following elements of $l^1(\mathbb{Z} \oplus \mathbb{Z}, \omega)$:

$$u = \delta_{(1,0)}, \qquad v = \delta_{(0,1)}.$$

Then $w_{(m,n)} = e^{imn\theta/2} u^m v^n$, hence

$$d_{(m,n)} = e^{imn\theta/2}(u^m v^n + u^{-m} v^{-n}).$$

It follows that

$$\begin{aligned} F(s,t) &= \sum_{m,n=-\infty}^{\infty} s^{|m|} t^{|n|} (u^m v^n + u^{-m} v^{-n}) = 2 \sum_{m,n=-\infty}^{\infty} s^{|m|} t^{|n|} u^m v^n \\ &= 2 \sum_{m=-\infty}^{\infty} s^{|m|} u^m \sum_{n=-\infty}^{\infty} t^{|n|} v^n. \end{aligned}$$

An elementary calculation shows that if z is any complex number having absolute value 1 and $-1 < s < 1$, then

$$\sum_{m=-\infty}^{\infty} s^{|m|} z^m = \frac{1-s^2}{1+s^2-s(z+\bar{z})} = \phi(s, z+\bar{z}).$$

Since $q = d_{(1,0)} = u + u^*$ and $p = d_{(0,1)} = v + v^*$, the assertion (i) follows from the analytic functional calculus.

To prove (ii), let A_{pq} be the coefficients in the power series expansion of F,

$$F(s,t) = \sum_{p,q=0}^{\infty} A_{pq} s^p t^q.$$

Obviously, $\{F(s,t) : s,t \in (-1,1)\}$ and $\{A_{pq} : p,q \geq 0\}$ have the same closed linear span. Using the fact that $d_{(-m,-n)} = d_{(m,n)}$, a straightforward computation shows that

$$A_{pq} = 2e^{-ipq\theta/2} d_{(p,q)} + 2e^{ipq\theta/2} d_{(-p,q)}.$$

Thus,

$$\begin{aligned} A_{0q} &= 2(d_{(0,q)} + d_{(0,q)}) = 4d_{(0,q)}, \qquad \text{and} \\ A_{p0} &= 2(d_{(p,0)} + d_{(-p,0)}) = 4d_{(p,0)}. \end{aligned}$$

In the remaining cases where $pq \neq 0$, the determinant of the coefficients of the 2×2 system of operator equations

$$\begin{aligned} A_{pq} &= 2e^{-ipq\theta/2} d_{(p,q)} + 2e^{ipq\theta/2} d_{(-p,q)}) \\ A_{pq}^* &= 2e^{ipq\theta/2} d_{(p,q)} + 2e^{-ipq\theta/2} d_{(-p,q)}) \end{aligned} \tag{4.7}$$

is $4(e^{-ipq\theta} - e^{ipq\theta}) \neq 0$, and in particular we can solve (4.7) for $d_{(p,q)}$ as a complex linear combination of A_{pq} and A_{pq}^*. This argument shows that the closed linear span of $\mathcal{F} \cup \mathcal{F}^*$ contains $\{d_{(p,q)} : p,q \in \mathbb{Z}\}$, and (ii) follows. That completes the proof of Theorem 4.2 □

Remark. In some very recent work [7], Bratteli and Kishimoto have established the striking result that $\mathcal{B}_\theta$ is an AF-algebra.

Appendix: Failure of Extensions. We present a simple example of a pair of commutative unital Banach $*$-algebras $B \subseteq A$ such that A is a subalgebra of its enveloping C^*-algebra, but such that the natural morphism $\gamma_B : C^*(B) \to C^*(A)$ is not injective. Let A be the algebra of all complex-valued continuous functions defined on the annulus $\{1 \leq |z| \leq 2\}$ which are analytic in its interior. With norm and involution defined by

$$\|f\| = \sup_{1 \leq |z| \leq 2} |f(z)|, \qquad f^*(z) = \bar{f}(\bar{z}),$$

$\bar{f}$ denoting the complex conjute of f, A is a unital Banach $*$-algebra. $C^*(A)$ is the commutative C^*-algebra $C(X)$,

$$X = [-2, -1] \cup [+1, +2]$$

denoting the intersection of the annulus $\{1 \leq |z| \leq 2\}$ with the real axis, and the completion map $\gamma : A \to C(X)$ is defined by restriction to X. Let B be the norm closure of all holomorphic polynomials in A. Then B is a self-adjoint subalgebra whose enveloping C^*-algebra is $C(Y)$, Y being the intersection of the polynomially convex hull of the annulus with the real axis, namely

$$Y = [-2, +2].$$

The morphism $\gamma_B : C(Y) \to C(X)$ is given by restriction to X, and hence there is a nontrivial kernel. Put differently, for every real $\lambda \in (-1, +1)$, the complex homomorphism of B defined by

$$\omega_\lambda(f) = f(\lambda), \qquad f \in B$$

is a bounded positive linear functional on B which cannot be extended to a positive linear functional on A.

References

1. Arveson, W., *Non-commutative spheres and numerical quantum mechanics*, preprint.
2. Avron, J., v. Mouche, P. H. M. and Simon, B, *On the measure of the spectrum for the almost Mathieu equation*, Comm. Math. Phys. **132** (1990), 103–118.
3. Bellissard, J.,Lima, R., and Testard, D., *On the spectrum of the almost Mathieu Hamiltonian*, preprint (1983).
4. Bellissard, J. and Simon, B., *Cantor spectrum for the almost Mathieu equation*, J. Funct. Anal. **48** (1982), 408–419.
5. Bratteli, O., Elliott, G., Evans, D. and Kishimoto, A., *Non-commutative spheres I*, preprint.
6. ———, *Non-commutative spheres II*, J. Op. Th. (to appear).
7. Bratteli, O. and Kishimoto, A., *Non-commutative spheres III*, manuscript.
8. Carmona, R. and Lacroix, J., *Spectral theory of random Schrödinger operators*, Birkhäuser, Boston, 1990.
9. Choi, M.-D. and Elliott, G., *Gauss polynomials and the rotation algebra*, Invent. Math. **99**, 225–246.
10. Connes, A., *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbb{R}$, Adv. Math. **39** (1981), 31–55.
11. Hulanicki, A., *On the symmetry of group algebras of discrete nilpotent groups*, Studia Math. **35** (1970), 207–219.
12. Jones, V. F. R., *Index for subfactors*, Inv. Math **72** (1983), 1–25.
13. Leptin, H., *Lokal Kompakte Gruppen mit Symmetrischen algebren*, Istituo Naz. di Alta Matematica, symposia mathematica **XXII** (1977).
14. Podleś, *Quantum spheres*, Letters in Math. Phys. **14** (1987), 193–203.
15. Rickart, C., *Banach Algebras*, van Nostrand, Princeton, 1960.
16. Riedel, N., *Point spectrum for the almost Mathieu equation*, C. R. Math. Rep. Acad. Sci. Canada VIII **6** (1986), 399-403.
17. Schweitzer, L., *Dense subalgebras of C^*-algebras with applications to spectral invariance*, Thesis, U.C. Berkeley (1991).