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The Universal A-Dynamical System

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ABSTRACT. For any C^*-algebra A, an A-dynamical system is a C^*-dynamical system that contains A and can be generated by the images of A under the semigroup of nonnegative time endomorphisms. There is a universal A-dynamical system that occupies a position in noncommutative dynamics that resembles the position of the tangent bundle in commutative dynamics.

We describe an approach to noncommutative dilation theory based on the universal A-dynamical system, emphasizing the role of continuous free products of C^*-algebras, noncommutative moment polynomials, and conditional expectations.

1. Introduction

This paper gives an exposition of a new approach to the dilation theory of semigroups of completely positive maps on von Neumann algebras. This approach is based on the notion of an A-dynamical system. These objects provide the C^*-algebraic structure that underlies much of noncommutative dynamics, whether it takes place in C^*-algebra or a von Neumann algebra, independently of issues relating to dilation theory. Indeed, for a fixed C^*-algebra A, the universal A-dynamical system occupies a position in noncommutative dynamics that is somewhat analogous to the position of the tangent bundle in commutative dynamics.

After describing the general properties of A-dynamical systems, we introduce α-expectations and noncommutative moment polynomials, and show how these objects enter into the construction of C^*-dilations. Indeed, once one is in possession of this C^*-algebraic infrastructure, a natural argument establishes the existence and uniqueness of dilations for quantum dynamical semigroups acting on von Neumann algebras. We give no proofs; but a technically complete and more comprehensive discussion can be found in Chapter 8 of the monograph [**Arv03b**]. While the exposition to follow has some overlap with Lecture 2 of [**Arv03a**], our objective here is to bring out the role of the universal A-dynamical system as a general tool in noncommutative dynamics.

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2. A-Dynamical Systems and Continuous Free Products

The flow of time in quantum theory is represented by a one-parameter group of $*$-automorphisms $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ of a C^*-algebra B. There is often a C^*-subalgebra $A \subseteq B$ that can be singled out from physical considerations which, together with its time translates, generates B. Borrowing a class of examples from [**Arv03b**], we recall that in a nonrelativistic quantum mechanical system with n degrees of freedom the flow of time is represented by a one-parameter group of automorphisms of $\mathcal{B}(L^2(\mathbb{R}^n))$ of the form $\alpha_t(T) = e^{itH}Te^{-itH}$, $t \in \mathbb{R}$, where H is a self-adjoint Schrödinger operator

$$H = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + V(X_1, \dots, X_n),$$

$X_1, \dots, X_n$ denote the configuration observables at time 0

$$X_k : \xi(x_1, \dots, x_n) \mapsto x_k \xi(x_1, \dots, x_n),$$

defined appropriately on a common dense domain in $L^2(\mathbb{R}^n)$, and V denotes the potential associated with the interaction forces. The functional calculus provides a faithful representation of the commutative C^*-algebra $C_0(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ by way of $f \mapsto f(X_1, \dots, X_n)$, and these functions of the configuration operators form a commutative C^*-subalgebra $A \subseteq \mathcal{B}(H)$. The family of commutative C^*-algebras $\{\alpha_t(A) : t \geq 0\}$ generates an irreducible C^*-subalgebra B of $\mathcal{B}(H)$; in particular, for different times $t_1 \neq t_2$, the C^*-algebras $\alpha_{t_1}(A)$ and $\alpha_{t_2}(A)$ fail to commute with each other. Indeed, there is no reason, physical or mathematical, to expect nontrivial relations to exist between the subalgebras $\alpha_{t_1}(A)$ and $\alpha_{t_2}(A)$ when $t_1 \neq t_2$.

We now examine this phenomenon in general. Throughout, A will denote an arbitrary but fixed C^*-algebra, with or without unit.

Definition 2.0.1. An A-dynamical system is a triple (ι, B, α) consisting of a semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of $*$-endomorphisms acting on a C^*-algebra B and an injective $*$-homomorphism $\iota : A \to B$, such that B is generated by $\cup_{t\geq 0}\alpha_t(\iota(A))$.

Note that we have imposed no continuity requirement on the semigroup α_t in its time parameter t. Given any C^*-algebra B containing A that is acted upon by a one-parameter group of $*$-automorphisms $\{\alpha_t : t \in \mathbb{R}\} \subseteq \operatorname{aut} B$, then there is an A-dynamical system associated with the C^*-subalgebra of B generated by A and its translates under α_t, $t \geq 0$. Conversely, every A-dynamical system (ι, B, α) is acted upon by a semigroup of $*$-endomorphisms, but in general these endomorphisms need not be extendable to automorphisms of a larger C^*-algebra containing B.

We identify A with its image $\iota(A)$ in B, thereby replacing ι with the inclusion map $A \subseteq B$. Thus, an A-dynamical system is a dynamical system (B, α) that contains A as a C^*-subalgebra in a specified way, with the property that B is the norm-closed linear span of finite products

$$B = \overline{\operatorname{span}}\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\cdots\alpha_{t_k}(a_k)\} \tag{2.1}$$

where $t_1, \dots, t_k \geq 0$, $a_1, \dots, a_k \in A$, $k = 1, 2, \dots$.

We will examine the class of *all* A-dynamical systems in order to secure more information about certain of its members. The above examples illustrate that even in cases where $A = C(X)$ is commutative, the structure of individual A-dynamical systems can be very complex.

There is a natural hierarchy in the class of all A-dynamical systems, defined by $(\iota, B, \alpha) \geq (\tilde{\iota}, \tilde{B}, \tilde{\alpha})$ iff there is a $*$-homomorphism $\theta : B \to \tilde{B}$ satisfying $\theta \circ \alpha_t = \tilde{\alpha}_t \circ \theta$, $t \geq 0$, and $\theta(a) = a$ for $a \in A$. Since θ fixes A, it follows from (2.1) that θ must be surjective, $\theta(B) = \tilde{B}$, hence $(\tilde{\iota}, \tilde{B}, \tilde{\alpha})$ is a *quotient* of (ι, B, α). Two A-dynamical systems are said to be *equivalent* if there is a map θ as above that is an isomorphism of C^*-algebras. This will be the case iff each of the A-dynamical systems dominates the other.

There is a largest equivalence class in this hierarchy, whose representatives are called *universal* A-dynamical systems. We will show that universal A-dynamical systems exist by first exhibiting the solution of a closely related universal problem.

Remark 2.0.2 (Noncommutative Path Space of a C^*-algebra). Consider the free product of an infinite family of copies of A indexed by the nonnegative reals

$$\mathcal{P}A = *_{t \geq 0} A_t, \qquad A_t = A.$$

By that we mean the following. We have a family of $*$-homomorphisms θ_t of A into a C^*-algebra $\mathcal{P}A$ such $\mathcal{P}A$ is generated by $\cup\{\theta_t(A) : t \geq 0\}$ and such that the following universal property is satisfied: for every family $\bar{\pi} = \{\pi_t : t \geq 0\}$ of $*$-homomorphisms of A into some other C^*-algebra B, there is a necessarily unique $*$-homomorphism $\rho : \mathcal{P}A \to B$ such that $\pi_t = \rho \circ \theta_t$, $t \geq 0$. Nondegenerate representations of $\mathcal{P}A$ correspond to families $\bar{\pi} = \{\pi_t : t \geq 0\}$ of representations $\pi_t : A \to \mathcal{B}(H)$ of A on a common Hilbert space H, subject to no condition other than the triviality of their common nullspace

$$\xi \in H, \quad \pi_t(A)\xi = \{0\} \quad \forall t \geq 0 \implies \xi = 0.$$

A simple argument establishes the existence of the continuous free product $\mathcal{P}A$ by taking the direct sum of a sufficiently large set of such representation families $\bar{\pi}$. The universal properties determine $\mathcal{P}A$ up to an obvious equivalence.

The functor $A \to \mathcal{P}A$ is a noncommutative replacement for the functor that replaces a space X with the space of all paths in X. Indeed, for every compact Hausdorff space X the cartesian product

$$PX = \prod \{X_t : t \geq 0\}, \qquad X_t = X, \quad t \geq 0,$$

is naturally a compact Hausdorff space whose points can be viewed as paths in X. Of course, individual "paths" can be very irregular functions, even non-measurable. $C(PX)$ is a commutative C^*-algebra with the property that for every $t \geq 0$, the map that evaluates an element of PX at time t defines an injective $*$-homomorphism $\theta_t : C(X) \to C(PX)$ by way of $\theta_t(f)(x) = f(x(t))$, $f \in C(X)$, $x \in PX$. These maps have the following universal property: For every commutative C^*-algebra B with unit and every family of unital $*$-homomorphisms $\pi_t : C(X) \to B$, there is a unique $*$-homomorphism $\rho : C(PX) \to B$ satisfying $\rho \circ \theta_t = \pi_t$, $t \geq 0$. Thus, *we pass from $C(PX)$ to $\mathcal{P}(C(X))$ by insisting that the universal property should persist for families of morphisms $\{\pi_t : t \geq 0\}$ of $C(X)$ to arbitrary C^*-algebras B.* The target C^*-algebras B are allowed to be noncommutative, nonunital, and the morphisms may be nonunital or even zero. The natural homomorphism of $\mathcal{P}C(X)$ on $C(PX)$ gives rise to an exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(X) \longrightarrow \mathcal{P}C(X) \longrightarrow C(PX) \longrightarrow 0$$

whose kernel $\mathcal{K}(X)$ appears as a somewhat mysterious object.

Finally, perhaps it is worth pointing out that when X is a smooth manifold, the tangent bundle TX of X can be viewed as an infinitesimal germ of the path space PX. Indeed, in this case we may consider the subspace $P^\infty X \subseteq PX$ of all smooth paths in X. For each $x \in P^\infty X$ we obtain a tangent vector at the point $x(0) \in X$ by differentiating the arc $t \mapsto x(t)$ in X at time zero. In this case the map $x \in P^\infty X \mapsto (x(0), \dot{x}(0)) \in TX$ defines a surjection of $P^\infty X$ onto TX.

We can now exhibit a universal A-dynamical system. The universal property of $\mathcal{P}A$ implies that there is a semigroup of shift endomorphisms $\sigma = \{\sigma_t : t \geq 0\}$ acting on $\mathcal{P}A$, defined uniquely by $\sigma_t \circ \theta_s = \theta_{t+s}$, $s, t \geq 0$. The same universal properties of $\mathcal{P}A$ lead easily to the fact that θ_0 is an injective $*$-homomorphism of A into $\mathcal{P}A$, and we use this map to identify A with $\theta_0(A) \subseteq \mathcal{P}A$. We may conclude that *the triple $(i, \mathcal{P}A, \sigma)$ becomes an A-dynamical system with the property that every other A-dynamical system is subordinate to it.*

Definition 2.0.3. $(i, \mathcal{P}A, \sigma)$ is called the universal A-dynamical system.

While this definition puts the the universal property of $\mathcal{P}A$ into the foreground, it fails to exhibit structural features of $\mathcal{P}A$ in terms that are concrete enough to establish its more subtle properties. We will present a more constructive definition of $\mathcal{P}A$ and the universal A-dynamical system in Section 5.

3. α-Expectations and C^*-Dilations

We will make essential use of conditional expectations onto hereditary subalgebras, and we begin by reviewing terminology. Given an inclusion of C^*-algebras $A \subseteq B$, a *conditional expectation* of B on A is an idempotent positive linear map $E : B \to A$ with range A, satisfying $E(ax) = aE(x)$ for $a \in A$, $x \in B$. Conditional expectations are completely positive linear maps of norm 1 whenever $A \neq \{0\}$. For any subset S of B we write $[S]$ for the norm-closed linear span of S. The subalgebra A is said to be *hereditary* if for $a \in A$ and $b \in B$, one has

$$0 \leq b \leq a \implies b \in A.$$

The hereditary subalgebra of B generated by a subalgebra A is the closed linear span $[ABA]$ of all products axb, $a, b \in A$, $x \in B$, and in general $A \subseteq [ABA]$. A *corner* of B is a hereditary subalgebra of the particular form $A = pBp$ where p is a projection in the multiplier algebra $M(B)$ of B.

When $A = pBp$ is a corner of B, the map $E(x) = pxp$, defines a conditional expectation of B onto A. On the other hand, many of the conditional expectations encountered here do not have this simple form, even when A has a unit. Indeed, if A is subalgebra of B that is *not* hereditary, then there is no natural conditional expectation $E : B \to A$. Most significantly for us, the universal A-dynamical system $(\iota, \mathcal{P}A, \sigma)$ never contains A as a hereditary subalgebra, hence there is no "obvious" conditional expectation $E : \mathcal{P}A \to A$.

Suppose now that we are given a semigroup $P = \{P_t : t \geq 0\}$ of completely positive contractions acting on a C^*-algebra A. We are interested in singling out certain A-dynamical systems (ι, B, α) with the property that there is a conditional expectation $E : B \to A$ with the property

$$E(\alpha_t(a)) = P_t(a), \qquad a \in A, \quad t \geq 0. \tag{3.1}$$

Of course, for many A-dynamical systems (ι, B, α) there will be no such conditional expectation; and even if such an expectation exists, there is no reason to expect it to be unique. What we require is the notion of an α-expectation.

Definition 3.0.4. Let (ι, B, α) be an A-dynamical system. An *α-expectation* is a conditional expectation $E : B \to A$ having the following two properties:

E1. Equivariance: $E \circ \alpha_t = E \circ \alpha_t \circ E$, $t \geq 0$.
E2. The restriction of E to the hereditary subalgebra generated by A is multiplicative, $E(xy) = E(x)E(y)$, $x, y \in [ABA]$.

Note that an *arbitrary* conditional expectation $E : B \to A$ gives rise to a family of linear maps $P = \{P_t : t \geq 0\}$ of A to itself by way of $P_t(a) = E(\alpha_t(a))$, $a \in A$. Each P_t is a completely positive contraction. When E is an α-expectation property E1 implies that P_t is related to α_t by

$$E \circ \alpha_t = P_t \circ E, \qquad t \geq 0. \tag{3.2}$$

The preceding formula (3.2) implies that P must satisfy the semigroup property $P_s \circ P_t = P_{s+t}$, as well as (3.1).

Property E2 is of course automatic if A is a hereditary subalgebra of B. It is a fundamentally *noncommutative* hypothesis on B. For example, if Y is a compact Hausdorff space and $B = C(Y)$, then every unital subalgebra $A \subseteq C(Y)$ generates $C(Y)$ as a hereditary algebra, and the only linear maps $E : C(Y) \to A$ satisfying E2 are $*$-endomorphisms of $C(Y)$.

Definition 3.0.5. Let $P = \{P_t : t \geq 0\}$ be a semigroup of completely positive contractions acting on a C^*-algebra A. A *C^*-dilation* of (A, P) is an A-dynamical system (ι, B, α) with the property that there is an α-expectation $E : B \to A$ satisfying (3.1):

$$P_t(a) = E(\alpha_t(a)), \qquad a \in A, \quad t \geq 0.$$

Notice that we have made no hypothesis of continuity of the semigroup P in its time variable. While the completely positive semigroups that arise in practice always obey some form of continuity in the time variable, it will be convenient to have the above flexibility. We will see in the following section that an α-expectation $E : B \to A$ is uniquely determined by the family of completely positive maps $\{P_t : t \geq 0\}$ defined by $P_t(a) = E(\alpha_t(a))$, $t \geq 0$; thus, the α-expectation associated with a C^*-dilation of a given CP semigroup P is *uniquely determined* by P.

It is a fundamental fact that C^*-dilations always exist. Indeed, the following result implies that the universal A-dynamical system is also a universal C^*-dilation in the sense that all semigroups of completely positive contractions that act on A can be dilated *simultaneously* to $(\iota, \mathcal{P}A, \sigma)$.

Theorem 3.0.6. *For every semigroup of completely positive contractions $P = \{P_t : t \geq 0\}$ acting on A, there is a unique σ-expectation $E : \mathcal{P}A \to A$ satisfying*

$$P_t(a) = E(\sigma_t(a)), \qquad a \in A, \quad t \geq 0. \tag{3.3}$$

Both assertions are nontrivial; we discuss uniqueness in the following section, existence is discussed in Section 5.

4. Moment Polynomials and n-point Functions

The theory of C^*-dilations rests on properties of certain noncommutative polynomials that are defined recursively as follows.

Proposition 4.0.7. *Let A be an algebra over a field $\mathbb{F}$. For every family of linear maps $\{P_t : t \geq 0\}$ of A to itself satisfying the semigroup property $P_{s+t} = P_s \circ P_t$ and $P_0 = \mathrm{id}$, there is a unique family of multilinear mappings from A to itself, indexed by the k-tuples of nonnegative real numbers, $k = 1, 2, \dots,$ where for a fixed k-tuple $\bar{t} = (t_1, \dots, t_k)$*

$$a_1, \dots, a_k \in A \mapsto [\bar{t}; a_1, \dots, a_k] \in A$$

is a k-linear mapping, all of which satisfy

MP1. $P_s([\bar{t}; a_1, \dots, a_k]) = [t_1 + s, t_2 + s, \dots, t_k + s; a_1, \dots, a_k]$.
MP2. *Given a k-tuple for which $t_\ell = 0$ for some ℓ between 1 and k,*

$$[\bar{t}; a_1, \dots, a_k] = [t_1, \dots, t_{\ell-1}; a_1, \dots, a_{\ell-1}] a_\ell [t_{\ell+1}, \dots, t_k; a_{\ell+1}, \dots, a_k].$$

The proofs of both existence and uniqueness are straightforward arguments using induction on the number k of variables. Note that in the second axiom MP2, we make the natural conventions when ℓ has one of the extreme values 1, k. For example, if $\ell = 1$, then MP2 should be interpreted as

$$[0, t_2, \dots, t_k; a_1, \dots, a_k] = a_1[t_2, \dots, t_k; a_2, \dots, a_k].$$

In particular, in the linear case $k = 1$, MP2 makes the assertion

$$[0; a] = a, \qquad a \in A;$$

and after applying axiom MP1 one obtains

$$[t; a] = P_t(a), \qquad a \in A, \quad t \geq 0,$$

thereby determining all moment polynomials of degree one.

One may calculate particular moment polynomials of higher degree explicitly, but the computations quickly become tedious. For example, in order to calculate $[6, 4, 2, 3; a, b, c, d] = P_2([4, 2, 0, 1; a, b, c, d])$, one writes

$$\begin{aligned}[4, 2, 0, 1; a, b, c, d] &= [4, 2; a, b]c[1, d] = [4, 2; a, b]cP_1(d) = P_2([2, 0; a, b])cP_1(d) \\ &= P_2([2, a]b)cP_1(d) = P_2(P_2(a)b)cP_1(d),\end{aligned}$$

and therefore

$$[6, 4, 2, 3; a, b, c, d] = P_2(P_2(P_2(a)b)cP_1(d)).$$

The computed value of $[t_1, \dots, t_k; a_1, \dots, a_k]$ depends strongly on the order relations that exist between the components of $t_1, \dots, t_k$. For example, after permuting the terms $6, 4, 2, 3$ of the previous example one finds that

$$[2, 6, 3, 4; a, b, c, d] = P_2(aP_1((P_3(b)cP_1(d))).$$

Finally, we remark that when A is a C^*-algebra and the linear maps satisfy $P_t(a)^* = P_t(a^*)$, $a \in A$, $t \geq 0$, then the associated moment polynomials obey the following symmetry

$$[t_1, \dots, t_k; a_1, \dots, a_k]^* = [t_k, \dots, t_1; a_k^*, \dots, a_1^*]. \tag{4.1}$$

Indeed, one simply notes that the sequence of polynomials $[[\cdot; \cdot]]$ defined by

$$[[t_1, \dots, t_k; a_1, \dots, a_k]] = [t_k, \dots, t_1; a_k^*, \dots, a_1^*]^*$$

also satisfies axioms MP1 and MP2, and hence must coincide with the moment polynomials of $\{P_t\}$ by the uniqueness assertion of Proposition 4.0.7.

Moment polynomials are important because they completely determine the expectation values of C^*-dilations of A in the following sense.

THEOREM 4.0.8. *Let $P = \{P_t : t \geq 0\}$ be a semigroup of completely positive maps on A satisfying $\|P_t\| \leq 1$, $t \geq 0$, with associated moment polynomials $[t_1, \dots, t_n; a_1, \dots, a_n]$.*

Let (i, B, α) be an A-dynamical system and let $E : B \to A$ be an α-expectation with the property $E(\alpha_t(a)) = P_t(a)$, $a \in A$, $t \geq 0$. Then the n-point functions of α are given by

$$E(\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\cdots\alpha_{t_n}(a_n)) = [t_1, \dots, t_n; a_1, \dots, a_n], \tag{4.2}$$

for every $n = 1, 2, \dots$, $t_i \geq 0$, $a_i \in A$. In particular, there is at most one α-expectation $E : B \to A$ satisfying $E(\alpha_t(a)) = P_t(a)$, $a \in A$, $t \geq 0$.

To summarize progress, we have seen that there is a universal A-dynamical system $(\iota, \mathcal{P}A, \sigma)$ such that every other A-dynamical system is a quotient of it. We have also singled out certain A-dynamical systems (ι, B, α) for which α expectations $E : B \to A$ exist, and for those A-dynamical systems we have seen that E is uniquely determined by the semigroup P of completely positive maps on A that it determines by way of

$$P_t(a) = E(\alpha_t(a)), \qquad t \geq 0, \quad a \in A.$$

Finally, starting with any semigroup of completely positive contractions $\{P_t : t \geq 0\}$ acting on A, we have seen by way of Theorem 3.0.6 that the universal A-dynamical system is actually a C^*-dilation of $\{P_t : t \geq 0\}$.

In the sections to follow we will say something about the proof of Theorem 3.0.6 and describe how that result leads to the appropriate dilation theorem for quantum dynamical semigroups acting on von Neumann algebras. We do not discuss the key issue of minimality in much detail here, but refer the reader to [**Arv03b**].

5. Construction of $\mathcal{P}A$ and C^*-dilations

Given a pair (A, P) consisting of a semigroup $P = \{P_t : t \geq 0\}$ of completely positive contractions on a C^*-algebra A, Theorem 3.0.6 asserts that there is a unique σ-expectation $E : \mathcal{P}A \to A$ satisfying

$$E(\sigma_t(a)) = P_t(a), \qquad t \geq 0, \quad a \in A.$$

Note that the uniqueness assertion follows from Theorem 4.0.8. The proof of existence is based on a construction that exhibits the structure of the continuous free product $\mathcal{P}A$ in concrete terms, along the following lines. One realizes $\mathcal{P}A$ as the enveloping C^*-algebra of a concrete Banach $*$-algebra $\ell^1(\Sigma)$ that is not a C^*-algebra, but that has appropriate universal properties. One is then able to write down a *natural* completely positive map on $\ell^1(\Sigma)$ that is associated with the moment polynomials of P; and the fact is that the latter map can be promoted through the completion procedure to obtain an σ-expectation on $\mathcal{P}A$ with the correct properties. We now describe this construction of $\mathcal{P}A$, omitting the proofs of key results (see Chapter 8 of [**Arv03b**]).

Let S be the set of finite sequences $\bar{t} = (t_1, t_2, \dots, t_k)$ of nonnegative real numbers t_i, $k = 1, 2, \dots$ which have distinct neighbors,

$$t_1 \neq t_2, t_2 \neq t_3, \dots, t_{k-1} \neq t_k.$$

Multiplication and involution are defined in S as follows. The product of two elements $\bar{s} = (s_1, \dots, s_k), \bar{t} = (t_1, \dots, t_\ell) \in S$ is defined by conditional concatenation

$$\bar{s} \cdot \bar{t} = \begin{cases} (s_1, \dots, s_k, t_1, \dots, t_\ell), & \text{if } s_k \neq t_1, \\ (s_1, \dots, s_k, t_2, \dots, t_\ell), & \text{if } s_k = t_1, \end{cases}$$

where we make the natural conventions when $\bar{t} = (t)$ is of length 1, namely $\bar{s} \cdot (t) = (s_1, \dots, s_k, t)$ if $s_k \neq t$, and $\bar{s} \cdot (t) = \bar{s}$ if $s_k = t$. The involution in S is defined by

$$(s_1, \dots, s_k)^* = (s_k, \dots, s_1).$$

One finds that S is an associative $*$-semigroup.

Fixing a C^*-algebra A, we attach a Banach space Σ_τ to every k-tuple $\tau = (t_1, \dots, t_k) \in S$ as follows

$$\Sigma_\tau = \underbrace{A \hat{\otimes} \cdots \hat{\otimes} A}_{k \text{ times}},$$

the k-fold projective tensor product of copies of the Banach space A. We assemble the various Σ_τ into a family $p : \Sigma \to S$ of Banach spaces over S by taking the total space to be $\Sigma = \{(\tau, \xi) : \tau \in S, \xi \in E_\tau\}$, with projection $p(\tau, \xi) = \tau$.

We introduce a multiplication in Σ as follows. Fix $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$ in S and choose $\xi \in \Sigma_\mu$, $\eta \in \Sigma_\nu$. If $\lambda_k \neq \mu_1$ then $\xi \cdot \eta$ is defined as the tensor product $\xi \otimes \eta \in \Sigma_{\mu \cdot \nu}$. If $\lambda_k = \mu_1$ then we must tensor over A and make the obvious identifications. More explicitly, in this case there is a natural map of the tensor product $\Sigma_\mu \otimes_A \Sigma_\nu$ onto $\Sigma_{\mu \cdot \nu}$ by making identifications of elementary tensors as follows:

$$(a_1 \otimes \cdots \otimes a_k) \otimes_A (b_1 \otimes \cdots \otimes b_\ell) \sim a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k b_1 \otimes b_2 \otimes \cdots \otimes b_\ell.$$

With this convention $\xi \cdot \eta$ is defined by

$$\xi \cdot \eta = \xi \otimes_A \eta \in \Sigma_{\mu \cdot \nu}.$$

This defines an associative multiplication in the family of Banach spaces Σ. There is also a natural involution in Σ, defined on each Σ_μ, $\mu = (s_1, \dots, s_k)$ as the unique antilinear isometry to Σ_{μ^*} satisfying

$$((s_1, \dots, s_k), a_1 \otimes \cdots \otimes a_k)^* = ((s_k, \dots, s_1), a_k^* \otimes \cdots \otimes a_1^*).$$

This defines an isometric antilinear mapping of the Banach space Σ_μ onto Σ_{μ^*}, for each $\mu \in S$, and thus the structure Σ becomes an involutive $*$-semigroup in which each fiber Σ_μ is a Banach space.

Let $\ell^1(\Sigma)$ be the Banach $*$-algebra of summable sections. The norm and involution are the natural ones $\|f\| = \sum_{\mu \in \Sigma} \|f(\mu)\|$, $f^*(\mu) = f(\mu^*)^*$. Noting that $\Sigma_\lambda \cdot \Sigma_\mu \subseteq \Sigma_{\lambda \cdot \mu}$, the multiplication in $\ell^1(\Sigma)$ is defined by convolution

$$f * g(\nu) = \sum_{\lambda \cdot \mu = \nu} f(\lambda) \cdot g(\mu),$$

and one easily verifies that $\ell^1(\Sigma)$ is a Banach $*$-algebra.

For $\mu = (s_1, \dots, s_k) \in S$ and $a_1, \dots, a_k \in A$ we define the function

$$\delta_\mu \cdot a_1 \otimes \cdots \otimes a_k \in \ell^1(\Sigma)$$

to be zero except at μ, and at μ it has the value $a_1 \otimes \cdots \otimes a_k \in \Sigma_\mu$. These elementary functions have $\ell^1(\Sigma)$ as their closed linear span. Finally, there is a natural family of $*$-homomorphisms $\theta_t : A \to \ell^1(\Sigma)$, $t \geq 0$, defined by

$$\theta_t(a) = \delta_{(t)} \cdot a, \qquad a \in A, \quad t \geq 0,$$

and these maps are related to the generating sections by

$$\delta_{(t_1,\dots,t_k)} \cdot a_1 \otimes \cdots \otimes a_k = \theta_{t_1}(a_1)\theta_{t_2}(a_2)\cdots\theta_{t_k}(a_k).$$

The algebra $\ell^1(\Sigma)$ fails to have a unit, but it has the same representation theory as $\mathcal{P}A$ in the following sense. Given a family of representations $\pi_t : A \to \mathcal{B}(H)$, $t \geq 0$, fix $\nu = (t_1, \dots, t_k) \in S$. There is a unique bounded linear operator $L_\nu : \Sigma_\nu \to \mathcal{B}(H)$ of norm 1 that is defined by its action on elementary tensors as follows

$$L_\nu(a_1 \otimes \cdots \otimes a_k) = \pi_{t_1}(a_1)\cdots\pi_{t_k}(a_k).$$

Thus there is a bounded linear map $\tilde{\pi} : \ell^1(\Sigma) \to \mathcal{B}(H)$ defined by

$$\tilde{\pi}(f) = \sum_{\mu \in S} L_\mu(f(\mu)), \qquad f \in \ell^1(\Sigma).$$

One finds that $\tilde{\pi}$ is a $*$-representation of $\ell^1(\Sigma)$ with $\|\tilde{\pi}\| = 1$. This representation satisfies $\tilde{\pi} \circ \theta_t = \pi_t$, $t \geq 0$. Conversely, every bounded $*$-representation $\tilde{\pi}$ of $\ell^1(\Sigma)$ on a Hilbert space H is associated with a family of representations π_t, $t \geq 0$, of A on H by way of $\pi_t = \tilde{\pi} \circ \theta_t$.

The results of the preceding discussion are summarized as follows:

Proposition 5.0.9. *The enveloping C^*-algebra $C^*(\ell^1(\Sigma))$, together with the family of homomorphisms $\tilde{\theta}_t : A \to C^*(\ell^1(\Sigma))$, $t \geq 0$, defined by promoting the homomorphisms $\theta_t : A \to \ell^1(\Sigma)$, has the same universal property as the infinite free product $\mathcal{P}A = *_{t \geq 0} a$, and is therefore isomorphic to $\mathcal{P}A$.*

There is a natural semigroup of $*$-endomorphisms of $\ell^1(\Sigma)$ defined by

$$\sigma_t : \delta_{(s_1,\dots,s_k)} \cdot \xi \mapsto \delta_{(s_1+t,\dots,s_k+t)} \cdot \xi, \qquad (s_1, \dots, s_k) \in \Sigma, \quad \xi \in \Sigma_\nu$$

and it promotes to the natural shift semigroup of $\mathcal{P}A = C^*(\ell^1(\Sigma))$. The inclusion of A in $\ell^1(\Sigma)$ is given by the map $\theta_0(a) = \delta_{(0)}a \in \ell^1(\Sigma)$, and it too promotes to the natural inclusion of A in $\mathcal{P}A$.

Finally, we fix a semigroup of completely positive contractions $P_t : A \to A$, $t \geq 0$, and consider the associated moment polynomials of Proposition 4.0.7. Since $\|P_t\| \leq 1$ for every $t \geq 0$, an inductive argument on the degree n shows that

$$\|[t_1, \dots, t_n; a_1, \dots, a_n]\| \leq \|a_1\| \cdots \|a_n\|, \qquad t_k \geq 0, \quad a_k \in A,$$

hence there is a unique bounded linear map $E_0 : \ell^1(\Sigma) \to A$ satisfying

$$E_0(\delta_{(t_1,\dots,t_k)} \cdot a_1 \otimes \cdots \otimes a_k) = [t_1, \dots, t_k; a_1, \dots, a_k],$$

for $(t_1, \dots, t_k) \in S$, $a_1, \dots, a_k \in A$, $k = 1, 2, \dots$, and in fact $\|E_0\| \leq 1$. Using the axioms MP1 and MP2, one finds that the map E_0 preserves the adjoint (see Equation (4.1)), satisfies the conditional expectation property $E_0(af) = aE_0(f)$ for $a \in A$, $f \in \ell^1(\Sigma)$, that the restriction of E_0 to the "hereditary" $*$-subalgebra of $\ell^1(\Sigma)$ spanned by $\theta_0(A)\ell^1(\Sigma)\theta_0(A)$ is multiplicative, and that it is related to ϕ by $E_0 \circ \sigma = \phi \circ E_0$ and $E_0(\sigma(a)) = \phi(a)$, $a \in A$. Thus, E_0 satisfies the axioms of Definition 3.0.4, suitably interpreted for the Banach $*$-algebra $\ell^1(\Sigma)$.

In view of the basic fact that a bounded completely positive linear map of a Banach $*$-algebra to A promotes naturally to a completely positive map of its enveloping C^*-algebra to A, the critical property of E_0 reduces to:

Theorem 5.0.10. *For every $n \geq 1$, $a_1, \dots, a_n \in A$, and $f_1, \dots, f_n \in \ell^1(\Sigma)$, we have*

$$\sum_{i,j=1}^{n} a_j^* E_0(f_j^* f_i) a_i \geq 0.$$

Consequently, E_0 extends uniquely through the completion map $\ell^1(\Sigma) \to \mathcal{P}A$ to a completely positive map $E : \mathcal{P}A \to A$ that becomes a σ-expectation satisfying Equation (3.3).

Corollary 5.0.11. *Every semigroup of completely positive contractions acting on a C^*-algebra has a C^*-dilation.*

6. Existence of W^*-dilations

We conclude by describing how one uses the results of the preceding sections to obtain dilations appropriate for the category of von Neumann algebras.

Definition 6.0.12. An *E-semigroup* is a semigroup $\{\alpha_t : t \geq 0\}$ of normal $*$-endomorphisms of a von Neumann algebra M that obeys the natural continuity requirement in its time variable, namely that for every normal linear functional ρ on M and $x \in M$, $\rho(\alpha_t(x))$ should move continuously in $t \in [0, \infty)$.

Let (M, α) be a pair consisting of a von Neumann algebra M with separable predual and an E-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on it. The operators $\alpha_t(\mathbf{1})$ form a decreasing family of projections in M in general, and if one has $\alpha_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$, then α is called an *E_0-semigroup*. The general issues discussed in this section do not depend on spatial aspects of M, and for the most part we will not have to realize M in any concrete representation as a subalgebra of $\mathcal{B}(H)$.

A *corner* of M is a von Neumann subalgebra of the particular form $N = pMp$, where p is a projection in M. The corner is said to be *full* if the central carrier of p is $\mathbf{1}$, and in that case pMp is a factor iff M is a factor of the same type.

Given an arbitrary projection $p \in M$, one can ask if there is a semigroup of completely positive maps $P = \{P_t : t \geq 0\}$ that acts on the corner pMp and is related to α as follows

$$P_t(pxp) = p\alpha_t(x)p, \qquad t \geq 0, \quad x \in M. \tag{6.1}$$

Such maps P_t need not exist in general; for example, taking $x = \mathbf{1} - p$, one finds that a necessary condition for P_t to exist is that p should satisfy $p\alpha_t(\mathbf{1} - p)p = 0$. Equivalently, a projection $p \in M$ is said to be *coinvariant* under α if

$$\alpha_t(\mathbf{1} - p) \leq \mathbf{1} - p, \qquad t \geq 0. \tag{6.2}$$

Remark 6.0.13 (Increasing Projections and E_0-semigroups). A projection $p \in M$ is said to be *increasing* if it has the property

$$\alpha_t(p) \geq p, \qquad t \geq 0. \tag{6.3}$$

Notice that in general, an increasing projection must be coinvariant. Indeed, since $\alpha_t(\mathbf{1}) \leq \mathbf{1}$, we will have

$$\alpha_t(\mathbf{1} - p) = \alpha_t(\mathbf{1}) - \alpha_t(p) \leq \mathbf{1} - p$$

whenever p is an increasing projection. The converse is not necessarily true. But in the special case where α is an E_0-semigroup, $\alpha_t(\mathbf{1} - p) = \mathbf{1} - \alpha_t(p)$; we conclude that *a projection is coinvariant under an E_0-semigroup iff it is an increasing projection.*

Now for any projection $p \in M$, one can define a family of linear maps $P = \{P_t : t \geq 0\}$ on $N = pMp$ by compressing each map α_t as follows

$$P_t(a) = p\alpha_t(a)p, \qquad a \in pMp, \quad t \geq 0. \tag{6.4}$$

Obviously, each P_t is a normal completely positive linear map of pMp into itself satisfying $\|P_t\| \leq 1$ for every $t \geq 0$. More significantly, one easily establishes:

PROPOSITION 6.0.14. *Let p be a coinvariant projection for α and consider the family of maps $P = \{P_t : t \geq 0\}$ of pMp defined by (6.4). P is a continuous semigroup of completely positive contractions, satisfying (6.1). If, in addition, α is an E_0-semigroup, then we have $P_t(p) = p$, $t \geq 0$.*

Dilation theory in the category of von Neumann algebras concerns the properties of completely positive semigroups that can be obtained from E-semigroups in this particular way. By a CP semigroup we mean a pair (N, P) where $P = \{P_t : t \geq 0\}$ is a semigroup of normal completely positive linear maps acting on a von Neumann algebra N which satisfies $\|P_t\| \leq 1$ for every $t \geq 0$.

DEFINITION 6.0.15. A triple (M, α, p) consisting of an E-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on a von Neumann algebra M, together with a distinguished coinvariant projection $p \in M$, is called a *dilation triple*. Let $N = pMp$ be the corner of M associated with p and let $P = \{P_t : t \geq 0\}$ be the semigroup acting on N as follows

$$P_t(a) = p\alpha_t(a)p, \qquad t \geq 0, \quad a \in N. \tag{6.5}$$

The CP semigroup (N, P) called a *compression* of (M, α, p), and (M, α, p) is called a *dilation* of (N, P).

We emphasize that the notion of a compression to a subalgebra has meaning only when (a) the subalgebra is a corner pMp of M and (b) the projection p satisfies (6.2). We have glossed over the key notion of *minimal* dilation. Indeed, there are several notions of minimality that are associated with this dilation theory, and it is a nontrivial fact that they are all equivalent, see [**Arv03b**]. In the discussion to follow we ignore considerations of minimality, confining attention to question of existence. However, we point out that an arbitrary dilation can always be reduced to a minimal one, and that a *necessary* condition for (M, α, p) to be a minimal dilation of (N, P) is that the central carrier of p should be $\mathbf{1}$. Thus, in the context of minimal dilations, if N is a factor then M must be a factor of the same type.

Starting with a CP semigroup (N, P), in order to find a dilation (M, α, p) of (N, P) one has to find a way of embedding N as a corner pMp of a larger von Neumann algebra M, on which there is a specified action of an E-semigroup α that is related to P as above. Notice that Corollary 5.0.11 provides the following infrastructure. If we view N as a unital C^*-algebra and P as a semigroup of contractive completely positive maps on N, forgetting the continuity of P_t in its time variable, then we can assert that the pair (N, P) has a C^*-dilation (ι, B, α). Certainly, B is not a von Neumann algebra and α is not an E-semigroup; thus (ι, B, α) cannot serve as a W^*-dilation of (N, P). However, it is possible to make use of the α-expectation $E : B \to N$ to find another dilation of (N, P) that is subordinate to (ι, B, α) and has all the desired properties. The results are summarized as follows.

THEOREM 6.0.16 (Existence of W^*-dilations). *Let $\{P_t : t \geq 0\}$ be a contractive CP-semigroup acting on a von Neumann algebra N with separable predual. Then (N, P) has a dilation (M, α, p).*

IDEA OF PROOF. Considering $P = \{P_t : t \geq 0\}$ as a semigroup of completely positive contractions acting on the unital C^*-algebra N, we see from Corollary 5.0.11 that P has a C^*-dilation (ι, B, α). We may obviously assume that $N \subseteq \mathcal{B}(H)$ acts concretely and nondegenerately on some separable Hilbert space H. We will construct a representation π of B on a Hilbert space $K \supseteq H$ with the property that each α_t can be extended to a normal $*$-endomorphism of the weak closure M of $\pi(B)$, and this will provide the required dilation of (N, P). The representation π is obtained as follows.

Let $E : B \to N$ be the α-expectation associated with (ι, B, α). Since we may view E as a completely positive map of B to $\mathcal{B}(H)$, it has a minimal Stinespring decomposition $E(x) = V^*\pi(x)V$, $x \in B$, where π is a representation of B on another Hilbert space K and $V : H \to K$ is a bounded linear map such that VH has K as its closed linear span.

Let M be the von Neumann algebra $\pi(B)''$. The remainder of the proof amounts to establishing three things. First, for every $t \geq 0$ there is a unique normal $*$-endomorphism $\tilde{\alpha}_t$ acting on M that satisfies $\tilde{\alpha}_t(\pi(x)) = \pi(\alpha_t(x))$, $x \in B$. Second, $\tilde{\alpha} = \{\tilde{\alpha}_t : t \geq 0\}$ is appropriately continuous in its time variable, thereby defining an E-semigroup acting on M. Third, that $p = VV^*$ is a projection in M whose corner pMp can be naturally identified with N, and that after this identification is made, $(M, \tilde{\alpha}, p)$ becomes a dilation triple for (N, P). □

Historical Remarks. A number of approaches to dilation theory for semigroups of completely positive maps have been proposed over the years, including work of Evans and Lewis [**EL77**], Accardi et al [**AL82**], Kümmerer [**Küm85**], Sauvageot [**Sau86**], and many others. Our attention was drawn to these developments by work of Bhat [**Bha99**], building on work of Bhat and Parthasarathy [**BP94**] for noncommutative Markov processes, in which the first dilation theory for CP semigroups acting on $\mathcal{B}(H)$ emerged that was effective for our work on E_0-semigroups [**Arv97**], [**Arv00**]. SeLegue [**SeL97**] showed how to apply multi-operator dilation theory to obtain Bhat's dilation result for CP semigroups acting on $\mathcal{B}(H)$, and he calculated the expectation values of the n-point functions of such dilations. Recently, Bhat and Skeide [**BS00**] have initiated an approach to the subject that is based on Hilbert modules over C^*-algebras and von Neumann algebras.

Special cases of Theorem 6.0.16 appeared in [**Bha99**], and a version of the full result appears in [**BS00**]. The approach taken here differs significantly from the former, and follows ideas from [**Arv02**], in which the dilation theory of a single completely positive map P on a C^*-algebra A is related directly to noncommutative dynamics by exploiting properties of the universal A-dynamical system, α-expectations, and the moment polynomials of P.

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